Ergodicity of canonical Gibbs measures with respect to the diffeomorphism group

Joint work with:

- T. Kuna
- Y. Kondratiev
- G. Goldin

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2. Canonical Gibbs measures
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Configuration spaces

- $X$ a connected, oriented Riemannian $l$-dimensional $C^\infty$ manifold with a volume element $m$ and metric $d$. The space of infinite configurations over $X$:

$$\Gamma_X := \{ \gamma \subset X | |\gamma \cap K| < \infty, K \subset X \text{ compact} \}$$
Configuration spaces

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- The space of n-point configurations in $Y \in \mathcal{B}(X)$

$$\Gamma_Y^{(n)} := \{ \eta \subset Y | |\eta| = n \}, \Gamma_Y^{(0)} := \{ \emptyset \}$$
Configuration spaces

- $X$ a connected, oriented Riemannian $l$-dimensional $C^\infty$ manifold with a volume element $m$ and metric $d$. The space of \textit{infinite configurations} over $X$:

$$\Gamma_X := \{ \gamma \subset X \mid |\gamma \cap K| < \infty, K \subset X \text{ compact} \}$$

- The space of $n$-point configurations in $Y \in \mathcal{B}(X)$

$$\Gamma_Y^{(n)} := \{ \eta \subset Y \mid |\eta| = n \}, \quad \Gamma_Y^{(0)} := \{ \emptyset \}$$

- We have:

$$\Gamma_X^{(n)} = \overline{X^n} / S_n$$

$$\overline{X^n} := \{ (x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ if } i \neq j \}$$
Gibbs measures

- Notation:

\[ \rho \in L^1_{\text{loc}}(X, m) \quad \text{m-a.s. strictly positive not necessary integrable} \]
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\[ \sigma := \rho m \]

The product \( \sigma^{\otimes n} \) has full measure on \( \widetilde{X^n} := \{(x_1, \ldots, x_n) \in X^n| x_i \neq x_j \text{ if } i \neq j\} \)

The symmetrization \( \sigma_n \) is a measure on \( \Gamma_0^{(n)} \)
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The symmetrization \( \sigma_n \) is a measure on \( \Gamma_0^{(n)} \)

• Lebesgue-Poisson measure on \( \Gamma_0 \)

\[ \lambda_{z\sigma} := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma_n \]
Poisson measure $\pi_{z\sigma}$ is the projective limit of

$$\pi_{z\sigma}^\Lambda := e^{-z\sigma(\Lambda)}\lambda_{z\sigma} \quad \Lambda \in \mathcal{O}_c(X)$$
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A measurable symmetric function

$$V : \widetilde{X}^2 \to \mathbb{R}$$

is called a pair potential
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Energy functional function

$$E : \Gamma_0 \to \mathbb{R}, \ \eta \mapsto \sum_{\{x,y\} \subset \eta} V(x, y), \ \text{and } E(\eta) := 0 \text{ for } |\eta| \leq 1.$$
Interaction energy:

\[ W(\eta, \gamma) := \begin{cases} 
\sum_{x \in \eta, y \in \gamma} V(x, y), & \text{if } \sum_{x \in \eta, y \in \gamma} |V(x, y)| < \infty \\
+\infty, & \text{otherwise}
\end{cases} \]
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+\infty, & \text{otherwise} 
\end{cases} \]

Conditional energy:

\[ E_\Lambda(\gamma) := E(\gamma_\Lambda) + W(\gamma_\Lambda, \gamma_{\Lambda^c}) \]
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Conditional energy:

\[ E_\Lambda(\gamma) := E(\gamma_\Lambda) + W(\gamma_\Lambda, \gamma_{\Lambda^c}) \]

Canonical specification:

\[ \Pi^c_\Lambda(F, \gamma) \]

\[ \frac{11\{0<Z_\Lambda<\infty\}(\gamma)}{Z_\Lambda(\gamma)} \int_{\Gamma_\Lambda^{(|\gamma_\Lambda|)}} 11_{F(\eta \cup \gamma_{\Lambda^c})} e^{-E(\eta)-W(\eta, \gamma_{\Lambda^c})} \sigma_{|\gamma_\Lambda|}(d\eta) \]
Definition (Gibbs measure)

A probability measure $\mu$ on $(\Gamma, \mathcal{B}(\Gamma))$ is called a canonical Gibbs measure iff

$$\mu \Pi^c_\Lambda = \mu, \ \forall \Lambda \in \mathcal{B}_c(X)$$

The set of all such measures is denoted by
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$$\mathcal{G}_c(V)$$

It has been shown in Preston’79 that $(\Pi^c_\Lambda)_{\Lambda \in \mathcal{B}_c(X)}$ is a $(\mathcal{F}^c_\Lambda)_{\Lambda \in \mathcal{B}_c(X)}$ - specification where

$$\mathcal{F}^c_\Lambda := \mathcal{F}^c_\Lambda(\Gamma) := \mathcal{B}_{X \setminus \Lambda}(\Gamma) \vee \sigma(\mathcal{N}_\Lambda^{-1}(\{n\}) | n \in \mathbb{N}_0),$$
K-transform

Functions on $\Gamma_0$

$$(K\ G)(\gamma) := \sum_{\xi \in \gamma} G(\xi)$$

$K$-transform

$\text{supp}(G) \subset \bigcup_{n=0}^{N} \Gamma^{(n)}_{\Lambda}$

Functions on $\Gamma$
K-transform

Functions on $\Gamma_0$

$K$-transform

$(KG)(\gamma) := \sum_{\xi \in \gamma} G(\xi)$

Functions on $\Gamma$

$\text{supp}(G) \subset \bigsqcup_{n=0}^{N} \Gamma_m^{(n)}$

• A probability measure $\mu$ on $(\Gamma, \mathcal{B}(\Gamma))$ has local finite moments: $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma)$ iff

$$\int_{\Gamma} |\gamma\Lambda|^n \mu(d\gamma) < \infty$$
K-transform

Functions on $\Gamma_0$ \quad Functions on $\Gamma$

\[ (KG)(\gamma) := \sum_{\xi \in \gamma} G(\xi) \quad \text{supp}(G) \subset \bigcup_{n=0}^{N} \Gamma^{(n)}_\Lambda \]

- A probability measure $\mu$ on $(\Gamma, \mathcal{B}(\Gamma))$ has local finite moments: $\mu \in \mathcal{M}^1_{\text{fm}}(\Gamma)$ iff
  \[ \int_{\Gamma} |\gamma\Lambda|^n \mu(d\gamma) < \infty \]

- The correlation measure corresponding to $\mu \in \mathcal{M}^1_{\text{fm}}(\Gamma)$
  \[ \int_{\Gamma_0} G(\eta) \rho_{\mu}(\eta) = \int_{\Gamma} (KG)(\gamma) \mu(\gamma) \]
• Let us denote the group of all diffeomorphisms

\[ \phi : X \rightarrow X \]

which are equal to identity outside of a compact set by 

\[ \text{Diff}_0(X) \]
Let us denote the group of all diffeomorphisms \( \phi : X \rightarrow X \) which are equal to identity outside of a compact set by \( \text{Diff}_0(X) \).

For technical reasons we have to work with a countable subgroup \( \text{Diff}_{\text{small}}(X) \) which still locally characterizes measures by their quasi-invariance.
More precisely, for any open connected set $O \in \mathcal{O}_c(X)$ and any measure $\tilde{\sigma}$ on $O$ which is quasi-invariant for all $\phi \in \text{Diff}_{\text{small}}(O)$ with the same Radon-Nikodym derivatives as $\sigma$, there exists $k > 0$ s.t.

$$\tilde{\sigma} = k\sigma$$
More precisely, for any open connected set $O \in \mathcal{O}_c(X)$ and any measure $\tilde{\sigma}$ on $O$ which is quasi-invariant for all $\phi \in \text{Diff}_{\text{small}}(O)$ with the same Radon-Nikodym derivatives as $\sigma$, there exists $k > 0$ s.t.

$$\tilde{\sigma} = k\sigma$$

The construction of such subgroup may be found in

- Gelfand, Graev abd Vershik’75
- Shimomura’95
Later we use the following lemma:

Lemma

Let $\mu$ be a prob. measure on $(\Gamma^{(n)}_\Lambda, \mathcal{B}(\Gamma^{(n)}_\Lambda))$, $\Lambda \in \mathcal{O}_c(X)$ measurable s.t. strictly positive $\mu \sim a.s.$

If $\mu$ is positive $\text{Diff}_{\text{small}}(X)$-quasi-invariant and

$$
\frac{d(\phi^* \mu)}{d\mu}(\eta) = \frac{r(\phi^{-1}(\eta))}{r(\eta)} \prod_{x \in \eta} J\phi^{-1}(x), \quad \forall \eta \in \Gamma^{(n)}_\Lambda
$$

$$
\Rightarrow \mu(d\eta) = kr(\eta) \lambda_m|_{\Gamma^{(n)}_\Lambda}(d\eta), \quad k > 0
$$
Assumptions on $V$ and $\mu$

Bounded below

$V(x, y) > -2B$, $B \geq 0$, $\forall x, y \in X$
Assumptions on $V$ and $\mu$

**Bounded below**

\[ V(x,y) > -2B, \quad B \geq 0, \quad \forall x, y \in X \]

**No hard core**

\[ \sup_{x,y\in X} V(x,y) < \infty, \quad \delta > 0 \]
Assumptions on $V$ and $\mu$

**Bounded below**

$$V(x, y) > -2B, \quad B \geq 0, \quad \forall x, y \in X$$

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$$\sup_{x, y \in X, d(x, y) > \delta} V(x, y) < \infty, \quad \delta > 0$$

**Regularity**

$$\int_X \left( \sup_{x \in \Lambda} |V(x, y)| \wedge 1 \right) \sigma(dy) < \infty, \quad \forall \Lambda \in O_c(X)$$
Assumptions on $V$ and $\mu$

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Regularity

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Admissible measure

$$\mu \in M^1_{fm}(\Gamma)$$

and the correlation measure $\rho_{\mu}^{(1)}$ is absolutely continuous w.r.t. $\sigma$ and we have

$$\frac{d\rho_{\mu}^{(1)}}{d\sigma}(x) \leq C_1 \quad C_1 > 0.$$
Corollary

For an admissible measure $\mu$ and $V$ satisfying the assumptions we have

$$\sum_{x \in \gamma \Lambda_c} \sup_{y \in \Lambda} |V(x, y)| < \infty, \quad \mu - a.a. \gamma \in \Gamma$$
Corollary

For an admissible measure $\mu$ and $V$ satisfying the assumptions we have

$$\sum_{x \in \gamma \Lambda} \sup_{y \in \Lambda} |V(x, y)| < \infty, \quad \mu - a.a. \gamma \in \Gamma$$

$$0 < Z_\Lambda(\gamma) < \infty \quad \mu - a.a. \gamma \in \Gamma$$
Corollary

For an admissible measure $\mu$ and $V$ satisfying the assumptions we have

$$\sum_{x \in \gamma, y \in \Lambda} \sup_{y \in \Lambda} |V(x, y)| < \infty, \quad \mu - a.a. \gamma \in \Gamma$$

$$0 < Z_\Lambda(\gamma) < \infty \quad \mu - a.a. \gamma \in \Gamma$$

For all $x \in \Lambda$ the sum is absolutely convergent

$$W(\{x\}, \gamma_c) = \sum_{y \in \gamma_c} V(x, y)$$
Characterization

We first derive the Radon-Nikodym for a canonical Gibbs measure w.r.t. the diffeomorphism group:
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**Theorem (Main result)**

Let $\mu \in \mathcal{M}_{\text{fm}}^1(\Gamma) \cap \mathcal{G}_c(V)$ be admissible.

Then $\mu$ is $\text{Diff}_0(X)$-quasi-invariant and

$$\frac{d(\phi^* \mu)}{d\mu}(\gamma) = \exp(-E_{\text{rel}}(\phi^{-1}(\gamma), \gamma)) \frac{d(\phi^* \pi_\sigma)}{d\pi_\sigma}(\gamma)$$
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**Theorem (Main result)**

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Then \( \mu \) is \( \text{Diff}_0(X) \)-quasi-invariant and

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\frac{d(\phi^{*}\mu)}{d\mu}(\gamma) = \exp (-E_{\text{rel}}(\phi^{-1}(\gamma), \gamma)) \frac{d(\phi^{*}\pi_{\sigma})}{d\pi_{\sigma}}(\gamma)
\]

Relative energy

\[
E_{\text{rel}}(\phi^{-1}(\gamma), \gamma) := \sum_{\{x,y\} \in \gamma} (V(\phi^{-1}(x), \phi^{-1}(y)) - V(x, y))
\]
Characterization

We first derive the Radon-Nikodym for a canonical Gibbs measure w.r.t. the diffeomorphism group:

Let \( \mu \in \mathcal{M}_{fm}^1(\Gamma) \cap \mathcal{G}_c(V) \) be admissible. Then \( \mu \) is \( \text{Diff}_0(X) \)-quasi-invariant and

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\frac{d(\phi^*\mu)}{d\mu}(\gamma) = \exp(-E_{\text{rel}}(\phi^{-1}(\gamma), \gamma)) \frac{d(\phi^*\pi_\sigma)}{d\pi_\sigma}(\gamma)
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E_{\text{rel}}(\phi^{-1}(\gamma), \gamma) := \sum_{\{x,y\} \in \gamma} (V(\phi^{-1}(x), \phi^{-1}(y)) - V(x, y))
\]
Characterization

**Proof:** Let $\phi \in \text{Diff}_0(X)$ and $F : \Gamma \to \mathbb{R}^+ \mathcal{B}_\Lambda(\Gamma)$-meas. Then we have:

$$\int_{\Gamma} F(\phi(\gamma))\mu(d\gamma) = \int_{\Gamma} \frac{\mathbb{1}_{\{0<Z_\Lambda<\infty\}}(\gamma)}{Z_\Lambda(\gamma)} \int_{\Gamma^{(1|\gamma_\Lambda|)}} F(\phi(\eta)) \times e^{-E(\eta)-W(\eta,\gamma_\Lambda c)} \sigma_{|\gamma_\Lambda|}(d\eta)\mu(\gamma)$$
Proof: Let $\phi \in \text{Diff}_0(X)$ and $F : \Gamma \to \mathbb{R}^+ \mathcal{B}_\Lambda(\Gamma)$ -meas. Then we have:

$$\int_{\Gamma} F(\phi(\gamma)) \mu(d\gamma) = \int_{\Gamma} \frac{1}{Z_\Lambda(\gamma)} \int_{\Gamma_{\Lambda^{|\gamma_\Lambda|}}} F(\phi(\eta)) \times e^{-E(\eta) - W(\eta, \gamma_{\Lambda^c})} \sigma_{|\gamma_{\Lambda}|}(d\eta) \mu(\gamma)$$

We note that:

$$\frac{d}{d \sigma_{|\gamma_{\Lambda}|}}(\eta) = \frac{d}{d \pi_{\sigma}}(\eta)$$

$$E_{\text{rel}}(\phi^{-1}(\eta \cup \gamma_{\Lambda^c}), \eta \cup \gamma_{\Lambda^c}) = E_\Lambda(\phi^{-1}(\gamma)) - E_\Lambda(\gamma),$$
Characterization

Therefore, applying the usual Radon-Nikodym theorem to the manifold \( \Gamma^{(|\gamma\Lambda|)}_{\Lambda} \) on the right hand side:

\[
\int_{\Gamma} \frac{\mathbb{1}_{\{0 < Z_{\Lambda} < \infty\}}(\gamma)}{Z_{\Lambda}(\gamma)} \int_{\Gamma^{(|\gamma\Lambda|)}_{\Lambda}} F\left(\phi(\eta)e^{-E(\eta) - W(\eta, \gamma_{\Lambda}^c)}\right)\sigma|_{\gamma_{\Lambda}}(d\eta)\mu(\gamma)
= \int_{\Gamma} \int_{\Gamma^{(|\gamma\Lambda|)}_{\Lambda}} F(\eta)e^{-E_{\text{rel}}(\phi^{-1}(\eta\cup\gamma_{\Lambda}^c), \eta\cup\gamma_{\Lambda}^c)} \frac{d(\phi^{*}\pi_{\sigma})}{d\pi_{\sigma}}(\eta)\Pi_{\Lambda}^{c}(d\eta, \gamma)\mu(d\gamma)
\]

The result follows by definition.
We proceed to show that
\[
\frac{d(\phi^* \mu)}{d\mu}(\gamma) = \exp(-E_{\text{rel}}(\phi^{-1}(\gamma), \gamma)) \frac{d(\phi^* \pi_\sigma)}{d\pi_\sigma}(\gamma)
\]
already characterizes canonical Gibbs measures.
Characterization

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\[ \frac{d(\phi^* \mu)}{d\mu}(\gamma) = \exp(-E_{\text{rel}}(\phi^{-1}(\gamma), \gamma)) \frac{d(\phi^* \pi_{\sigma})}{d\pi_{\sigma}}(\gamma) \]

already characterizes canonical Gibbs measures

Theorem

Let \( \mu \in M^1_{\text{fm}}(\Gamma) \) be an admissible measure and the potential fulfills the assumptions. If for all \( \phi \in \text{Diff}_{\text{small}}(X) \) we have

\[ \frac{d(\phi^* \mu)}{d\mu}(\gamma) = \exp(-E_{\text{rel}}(\phi^{-1}(\gamma), \gamma)) \frac{d(\phi^* \pi_{\sigma})}{d\pi_{\sigma}}(\gamma) \]

\[ \implies \mu \in G_c(V) \]
Proof: The relative energy is well defined according the corollary above: For $\phi \in \text{Diff}_{\text{small}}(X)$ s.t. $\text{supp}(\phi) \subset \Lambda$

Take:

$$F = F_1 \cdot F_2 \quad \text{where} \quad F_1 \in L^0(\Gamma, B_\Lambda(\Gamma)) \quad \text{and} \quad F_2 \in L^0(\Gamma, B_{\Lambda^c}(\Gamma))$$
**Proof:** The relative energy is well defined according the corollary above: For $\phi \in \text{Diff}_{\text{small}}(X)$ s.t. $\text{supp}(\phi) \subset \Lambda$

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Hence, using the definition of conditional probability we can write

$$ \int_{\Gamma} F(\phi(\gamma)) \mu(d\gamma) = \int_{\Gamma} \int_{\Gamma^{(\gamma \Lambda)}} F_2(\gamma) F_1(\phi(\eta)) \mu_\Lambda(d\eta, \gamma) \mu(d\gamma) $$
**Proof:** The relative energy is well defined according the corollary above: For $\phi \in \text{Diff}_{\text{small}}(X)$ s.t. $\text{supp}(\phi) \subset \Lambda$

Take:

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Hence, using the definition of conditional probability we can write

$$\int_{\Gamma} F(\phi(\gamma)) \mu(d\gamma) = \int_{\Gamma} \int_{\Gamma^{(\gamma \Lambda^1)}} F_2(\gamma) F_1(\phi(\eta)) \mu_\Lambda(d\eta, \gamma) \mu(d\gamma)$$

On the other hand we have

$$\int_{\Gamma} F(\phi(\gamma)) \mu(d\gamma) = \int_{\Gamma} F_2(\gamma) \int_{\Gamma^{(\gamma \Lambda^1)}} F_1(\eta) e^{-E_{\text{rel}}(\phi^{-1}(\eta) \cup \gamma_{\Lambda^c}, \eta \cup \gamma_{\Lambda^c})}$$

$$\times \frac{d(\phi^* \pi_\sigma)}{d\pi_\sigma}(\eta) \mu_\Lambda(d\eta, \gamma) \mu(d\gamma).$$
Because of the countability of \( \text{Diff}_{\text{small}}(X) \) for \( \mu \)-a.a. \( \gamma \)

\[
\int_{\Gamma_{\Lambda}^{(|\gamma_{\Lambda}|)}} F_3(\phi(\eta)) \mu_{\Lambda}(d\eta, \gamma) = \int_{\Gamma_{\Lambda}^{(|\gamma_{\Lambda}|)}} F_3(\eta) e^{-E_{\text{rel}}(\phi^{-1}(\eta) \cup \gamma_{\Lambda}^c, \eta \cup \gamma_{\Lambda}^c)} \times \frac{d(\phi^* \pi_{\sigma})}{d\pi_{\sigma}}(\eta) \mu_{\Lambda}(d\eta, \gamma)
\]

for any \( F_3 \in L^0(\Gamma, \mathcal{B}(\Gamma_{\Lambda})) \)
Because of the countability of \( \text{Diff}_{\text{small}}(X) \) for \( \mu \)-a.a. \( \gamma \)

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\]

\[
\times \frac{d(\phi^*\pi_\sigma)}{d\pi_\sigma}(\eta)\mu_\Lambda(d\eta, \gamma)
\]

for any \( F_3 \in L^0(\Gamma, \mathcal{B}(\Gamma_{\Lambda})) \)

The corollary and lemma above now implies that the conditional probability coincides with the canonical specification, i.e., \( \Pi^c_\Lambda = \mu_\Lambda \). Hence \( \mu \in \mathcal{G}_c(V) \)
Ergodicity

Definition

- A measure $\mu$ on $\Gamma$ is said \textit{Diff}_0(X)-ergodic if the $\mu$-a.s. constant functions are the only bounded measurable functions $F : \Gamma \to \mathbb{R}^+$ which have the property $F \circ \phi = F$ $\mu$-a.s. for all $\phi \in \text{Diff}_0(X)$.
**Ergodicity**

**Definition**

- A measure $\mu$ on $\Gamma$ is said to be **Diff$_0$(X)-ergodic** if the $\mu$-a.s. constant functions are the only bounded measurable functions $F : \Gamma \to \mathbb{R}^+$ which have the property $F \circ \phi = F$ $\mu$-a.s. for all $\phi \in \text{Diff}_0(X)$.

- A measure $\mu$ from the convex set $\mathcal{G}_c(V)$ is said to be **extreme** if $\forall \mu_1, \mu_2 \in \mathcal{G}_c(V)$

$$\mu = \alpha \mu_1 + (1 - \alpha) \mu_2 \implies \mu = \mu_1 = \mu_2$$
Ergodicity

Definition

• A measure $\mu$ on $\Gamma$ is said $\text{Diff}_0(X)$-ergodic if the $\mu$-a.s. constant functions are the only bounded measurable functions $F : \Gamma \to \mathbb{R}^+$ which have the property $F \circ \phi = F$ $\mu$-a.s. for all $\phi \in \text{Diff}_0(X)$

• A measure $\mu$ from the convex set $\mathcal{G}_c(V)$ is said to be extreme: $\forall \mu_1, \mu_2 \in \mathcal{G}_c(V)$

$$\mu = \alpha \mu_1 + (1 - \alpha) \mu_2 \implies \mu = \mu_1 = \mu_2$$

• The tail field $\sigma$-algebra is defined as

$$\mathcal{F}_\infty(\Gamma) := \bigcap_{\Lambda \in \mathcal{B}_c(X)} \mathcal{F}_{\Lambda^c}(\Gamma)$$
Ergodicity

Lemma [Preston’76]

- \( \mu, \mu' \in G_c(V) \) and \( F \) positive s.t.

\[
\int_{\Gamma} F(\gamma) \mu(d\gamma) = 1
\]
Ergodicity

Lemma [Preston’76]

• $\mu, \mu' \in G_c(V)$ and $F$ positive s.t.

\[ \int_{\Gamma} F(\gamma) \mu(d\gamma) = 1 \]

- $\mu$ is extreme iff it is trivial in $F_{\infty}(\Gamma)$, i.e.

\[ \mu(B) = \begin{cases} 0 & \text{for } B = 0 \\ 1 & \text{for } B = 1 \end{cases} \]
Ergodicity

Lemma [Preston’76]

• \( \mu, \mu' \in \mathcal{G}_c(V) \) and \( F \) positive s.t.

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\int_{\Gamma} F(\gamma) \mu(d\gamma) = 1
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▶ \( \mu \) is extreme iff it is trivial in \( \mathcal{F}_\infty(\Gamma) \), i.e.

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\mu(B) = \begin{cases} 
0 & \text{for each } B \in \mathcal{F}_\infty(\Gamma) \\
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\]

▶ \( F\mu \in \mathcal{G}_c(V) \iff \mathbb{E}_\mu [F|\mathcal{F}_\infty(\Gamma)] = F\mu \text{--a.s.} \)
Ergodicity

Lemma [Preston’76]

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  \[
  \int_\Gamma F(\gamma)\mu(d\gamma) = 1
  \]

- \( \mu \) is extreme iff it is trivial in \( \mathcal{F}_\infty(\Gamma) \), i.e.
  \[
  \mu(B) = \begin{cases} 
  0 & \text{for each } B \in \mathcal{F}_\infty(\Gamma) \\
  1 &
  \end{cases}
  \]

- \( F\mu \in \mathcal{G}_c(V) \iff \mathbb{E}_\mu[F|\mathcal{F}_\infty(\Gamma)] = F\mu - \text{a.s.} \)

- \( \mu \neq \mu' \implies \mu \perp \mu' \iff \exists B \in \mathcal{F}_\infty(\Gamma) : \mu(B) = 1 \land \mu'(B) = 0. \)
Ergodicity

Lemma

• Let \( \mu \in M^1_{\text{fm}}(\Gamma) \cap G_c(V) \) be admissible and \( F \) with \[
\int_{\Gamma} F(\gamma) \mu(d\gamma) = 1
\]

and

\[
F \circ \phi = F \mu - \text{a.s. for all } \phi \in \text{Diff}_{\text{small}}(X)
\]

• Then \( \nu := F \mu \) is also a canonical Gibbs measure.

Proof: for another positive meas. function \( G : \Gamma \to \mathbb{R}^+ \)

\[
\int_{\Gamma} G(\phi(\gamma)) \nu(d\gamma) = \int_{\Gamma} G(\gamma) F(\phi^{-1}(\gamma)) d(\phi^* \mu)(\gamma) = \int_{\Gamma} G(\gamma) \frac{d(\phi^* \mu)}{d\mu}(\gamma) \nu(d\gamma)
\]
Ergodicity

\[ \frac{d(\phi^* \mu)}{d\mu}(\gamma) = \frac{d(\phi^* \nu)}{d\nu}(\gamma), \quad \nu-a.s.\]
Ergodicity

\[ \frac{d(\phi^* \mu)}{d\mu}(\gamma) = \frac{d(\phi^* \nu)}{d\nu}(\gamma), \nu-a.s. \]

Furthermore, for \( H : \Gamma_0 \to \mathbb{R}^+ \)

\[ \int_{\Gamma_0} H(\eta) \rho_\nu(d\eta) = \int_{\Gamma} (KH)(\gamma) F(\gamma) \mu(d\gamma) \leq C \int_{\Gamma} (KH)(\gamma) \mu(d\gamma) \]
Ergodicity

\[ \frac{d(\phi^* \mu)}{d\mu}(\gamma) = \frac{d(\phi^* \nu)}{d\nu}(\gamma), \ \nu - \text{a.s.} \]

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\[
\int_{\Gamma_0} H(\eta) \rho_\nu(d\eta) = \int_{\Gamma} (KH)(\gamma) F(\gamma) \mu(d\gamma) \leq \sup_{\gamma \in \Gamma} |F(\gamma)| \]

\[
\leq C \int_{\Gamma} (KH)(\gamma) \mu(d\gamma)
\]
Ergodicity

\[ d(\phi^* \mu) \frac{d(\phi^* \mu)}{d\mu} (\gamma) = d(\phi^* \nu) \frac{d(\phi^* \nu)}{d\nu} (\gamma), \ \nu - \text{a.s.} \]

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\[ \Rightarrow \rho_\nu (d\eta) \leq C \rho_\mu (d\eta) \]
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\[ \frac{d(\phi^* \mu)}{d\mu}(\gamma) = \frac{d(\phi^* \nu)}{d\nu}(\gamma), \ \nu - a.s. \]

Furthermore, for \( H : \Gamma_0 \to \mathbb{R}^+ \)

\[ \int_{\Gamma_0} H(\eta) \rho_\nu(\eta) \, d\eta = \int_\Gamma (K \, H)(\gamma) F(\gamma) \mu(d\gamma) \leq C \int_\Gamma (K \, H)(\gamma) \mu(d\gamma) \]

\[ \implies \rho_\nu(\eta) \leq C \rho_\mu(\eta) \]

Hence \( \mu \) and \( \nu \) fulfills the assumptions of characterization theorem via Radon-Nikodym derivatives and we deduce \( \nu \in G_c(V) \)
Theorem (Main result)

An admissible canonical Gibbs measure is extreme iff it is ergodic w.r.t. $\text{Diff}_{\text{small}}(X)$

$$\mu \in \text{ext}(\mathcal{G}_{c,a}(V)) \iff \text{Diff}_{\text{small}}(X)\text{--ergodic}$$
**Ergodicity**

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**Proof:** Assume that $\mu$ is extreme: let $F : \Gamma \rightarrow \mathbb{R}^+$ be meas. bounded s.t. $F \circ \phi = F$ for all $\phi \in \text{Diff}_0(X) \mu$-a.s.
Ergodicity

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According to the previous lemmas:

\[ \implies F\mu \in \mathcal{G}_c(V) \]
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According to the previous lemmas:

$$\implies F\mu \in \mathcal{G}_c(V)$$

$$\implies \mathbb{E}_\mu(F|\mathcal{F}_{\infty}(\Gamma)) = F, \ \mu - \text{a.s.}.$$
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$$\implies \mu \text{ is trivial on } \mathcal{F}_\infty(\Gamma)$$

$$\implies \mu \text{ is } \text{Diff}_{\text{small}}(X) - \text{ergodic}$$
Conversely, assume that $\mu$ is $\text{Diff}_{\text{small}}(X)-\text{ergodic}$ and

$$\mu = \frac{1}{2}(\mu_1 + \mu_2)$$

Thus $\mu_1 \ll \mu$ and there exists a measurable function $F : \Gamma \to \mathbb{R}^+$, 

$$\int_{\Gamma} F(\gamma) \mu(d\gamma) = 1, \quad \mu_1 = F\mu$$
Conversely, assume that $\mu$ is $\text{Diff}_{\text{small}}(X)$–ergodic and

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**lemma** $\Rightarrow \mathbb{E}_\mu(F|\mathcal{F}_\infty(\Gamma)) = F\mu - \text{a.s.}$
Ergodicity

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$$F : \Gamma \to \mathbb{R}^+, \quad \int_{\Gamma} F(\gamma) \mu(d\gamma) = 1, \quad \mu_1 = F\mu$$

**Lemma**  \[ \Rightarrow \mathbb{E}_\mu(F|\mathcal{F}_\infty(\Gamma)) = F \mu - \text{a.s.} \]

Hence for any $\phi \in \text{Diff}_0(X)$

$$F \circ \phi = \mathbb{E}_\mu(F|\mathcal{F}_\infty(\Gamma)) \circ \phi = \mathbb{E}_\mu(F|\mathcal{F}_\infty(\Gamma)) = F.$$
Conversely, assume that \( \mu \) is \( \text{Diff}_{\text{small}}(X) \)-ergodic and
\[
\mu = \frac{1}{2}(\mu_1 + \mu_2)
\]
Thus \( \mu_1 \ll \mu \) and there exists a measurable function
\[
F : \Gamma \to \mathbb{R}^+, \quad \int_{\Gamma} F(\gamma) \mu(d\gamma) = 1, \quad \mu_1 = F\mu
\]

\[
\text{lemma} \quad \implies \mathbb{E}_\mu(F|\mathcal{F}_\infty(\Gamma)) = F \mu - a.s.
\]

Hence for any \( \phi \in \text{Diff}_0(X) \)
\[
F \circ \phi = \mathbb{E}_\mu(F|\mathcal{F}_\infty(\Gamma)) \circ \phi = \mathbb{E}_\mu(F|\mathcal{F}_\infty(\Gamma)) = F.
\]
\[
\implies F \text{ is constant } \mu \text{-a.s.} \quad \implies \mu = \mu_1 = \mu_2
\]
Ergodicity

Theorem

Let $\mu$ be an admissible canonical Gibbs measure. Then the unitary representation

$$(V_{\mu}(\phi)F)(\gamma) := \sqrt[\gamma]{ \frac{d\phi^*\mu}{d\mu} (\gamma) F(\phi^{-1}(\gamma))}, \quad F \in L^2(\Gamma, \mu)$$

is irreducible iff $\mu$ is extreme.
Ergodicity

Theorem

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$$(V_\mu(\phi)F)(\gamma) := \sqrt{\frac{d\phi^*\mu}{d\mu}}(\gamma)F(\phi^{-1}(\gamma)), \quad F \in L^2(\Gamma, \mu)$$

is irreducible iff $\mu$ is extreme.

Proof: Consequence of the results of this section and

• Theorem 1 of Gelfand, Graev abd Vershik’75
• or Corollary 28.1 of Ismagilov’96
Under weak assumptions on the potential we have proved that the canonical Gibbs measures are characterized by their Radon-Nikodym derivatives.
Conclusions and generalizations

Conclusions

- Under weak assumptions on the potential we have proved that the canonical Gibbs measures are characterized by their Radon-Nikodym derivatives.

- We used conditional expectations to reduce ourselves to finite configurations. The absence of coinciding points allowed us to reduce further to quasi-invariant measures on open subsets of $\mathbb{R}^l$. 
**Conclusions**

If \( \mu \in M^1_{fm}(\Gamma) \cap G_c(V) \) is admissible and \( V \) fulfills assumptions then

\[
\mu \text{ is extreme } \iff \text{Diff}_{\text{small}-\text{ergodic}}
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Conclusions and generalizations

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- If $\mu \in \mathcal{M}^1_{fm}(\Gamma) \cap \mathcal{G}_c(V)$ is admissible and $V$ fulfills assumptions then

  $\mu$ is extreme $\iff$ Diff$_{\text{small}}$–ergodic

- Then the unitary representation

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Conclusions and generalizations

Generalizations

1. Marked systems: $X \times S$
Conclusions and generalizations

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Another manifold

internal degree of freedom:

★ momentum, spin
★ charge, dipole-moment
Conclusions and generalizations

Generalizations

1. Marked systems: $X \times S$

Another manifold

- Internal degree of freedom:
  - Momentum, spin
  - Charge, dipole-moment

• Configurations: $\hat{\gamma} \in \Gamma_{X \times S}$

$(x, s), (y, t) \in \hat{\gamma}$

$(x, s) \neq (y, t) \implies x \neq y$
Conclusions and generalizations

Generalizations

- Diffeomorphism: \( \hat{\phi} \in \text{Diff}_0(X \times S) \)

\[
\hat{\phi}(x, s) := (\phi(x), \psi(x, s))
\]

If we assume that the marked space is a Lie group, then we may assume that \( \psi(x, \cdot) \) is from the structure group:

\[
\psi(x, s) = \tilde{\psi}(x) \cdot s, \quad \tilde{\psi} : X \to S
\]
Generalizations

Then we showed, following the same line of proof as before, that it is essential to have the characterization by Radon-Nikodym derivatives for measures on the one particle space $X \times S$ to derive the characterization result for canonical Gibbs measures.
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The extremal canonical Gibbs measures are ergodic w.r.t. the considered subgroup of diffeomorphisms.
Generalizations

- Then we showed, following the same line of proof as before, that it is essential to have the characterization by Radon-Nikodym derivatives for measures on the one particle space $X \times S$ to derive the characterization result for canonical Gibbs measures.

- The extremal canonical Gibbs measures are ergodic w.r.t. the considered subgroup of diffeomorphisms.

- However, in general the corresponding representations on the corresponding $L^2$-spaces will be not any longer irreducible.
Conclusions and generalizations

Generalizations

2. General interactions than pair potentials:

Consider functions: \( V : \Gamma_0 \rightarrow \mathbb{R} \) and define the corresponding conditional energy:

\[
E_\Lambda(\eta \cup \gamma_\Lambda^c) := \sum_{\eta' \subset \eta} \sum_{\xi \in \gamma_\Lambda^c} V(\eta' \cup \xi),
\]

if the series is absolutely convergent and by infinity otherwise.
Generalizations

2. **General interactions** than pair potentials:

Consider functions: \( V : \Gamma_0 \to \mathbb{R} \) and define the corresponding conditional energy:

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E_{\Lambda}(\eta \cup \gamma_{\Lambda^c}) := \sum_{\eta' \subset \eta} \sum_{\xi \in \gamma_{\Lambda^c}} V(\eta' \cup \xi),
\]

if the series is absolutely convergent and by infinity otherwise.

- The main technical difficulty is to give concrete conditions on the measure and potential such that uniformly the convergence of the series may be controlled!
Conclusions and generalizations

Generalizations

Assumption:

- Let \( \mu \in \mathcal{M}_{\text{fm}}^1(\Gamma) \) be given and \( \rho_\mu \) the corresponding correlation measure. Assume:

- \( \rho_\mu \ll \lambda_\sigma \), \( V_0 : \Gamma \to \mathbb{R} \) continuous
Conclusions and generalizations

Generalizations

Assumption:

• Let $\mu \in \mathcal{M}^1_{fm}(\Gamma)$ be given and $\rho_\mu$ the corresponding correlation measure. Assume:

  • $\rho_\mu \ll \lambda_\sigma$, $V_0 : \Gamma \to \mathbb{R}$ continuous

  • $\int_{\Gamma^{\Lambda_c}_c} \left( \sup_{\eta \in \Gamma^{(n)}_{\delta, \Lambda}} |V(\eta \cup \xi)| \wedge 1 \right) \rho_\mu(d\xi) < \infty$

\[
\begin{align*}
\delta &= \frac{1}{m} \\
\gamma &= \frac{1}{n}
\end{align*}
\]
Conclusions and generalizations

Generalizations

Assumption:

- Let $\mu \in \mathcal{M}^1_{fm}(\Gamma)$ be given and $\rho_\mu$ the corresponding correlation measure. Assume:
  
  - $\rho_\mu \ll \lambda_\sigma$, \quad $V_0 : \Gamma \rightarrow \mathbb{R}$ continuous
  
  - $\int_{\Gamma_{\Lambda c}} \left( \sup_{\eta \in \Gamma^{(n)}_{\delta, \Lambda}} |V(\eta \cup \xi)| \wedge 1 \right) \rho_\mu(d\xi) < \infty$

  $\Gamma_{\delta, \Lambda}^{(n)} := \left\{ \eta \in \Gamma_{\Lambda}^{(n)} \mid d(x, y) > \delta \text{ for all } \{x, y\} \subset \eta \right\}$
Conclusions and generalizations

Generalizations

Under this assumption the techniques from before show that for $\mu$-a.a. $\gamma \in \Gamma$ the series is uniformly convergent for all $\eta \in \Gamma^{(n)}_{\Lambda}$

$$E_{\Lambda}(\eta \cup \gamma_{\Lambda}^{c}) := \sum \sum V(\eta' \cup \xi),$$

with $\eta' \subset \eta$, $\xi \in \gamma_{\Lambda}^{c}$, $\eta' \neq \emptyset$.
Conclusions and generalizations

Generalizations

Under this assumption the techniques from before show that for $\mu$-a.a. $\gamma \in \Gamma$ the series is uniformly convergent for all $\eta \in \Gamma^{(n)}_{\Lambda}$. 

$$E_{\Lambda}(\eta \cup \gamma_{\Lambda^c}) := \sum_{\eta' \subset \eta, \eta' \neq \emptyset} \sum_{\xi \subseteq \gamma_{\Lambda^c}} V(\eta' \cup \xi),$$

Theorem

Let $\mu$ be a measure which fulfills the assumption above. Then

$$\mu \in G_{c}(V) \iff \mu \text{ is } \text{Diff}_{0}(X)-\text{quasi}-\text{invariant}$$

with Radon-Nikodym derivative:
Conclusions and generalizations

Generalizations

\[
\frac{d(\phi^* \mu)}{d\mu}(\gamma) = \exp(-E_{\text{rel}}(\phi^{-1}(\gamma), \gamma)) \frac{d(\phi^* \pi_\sigma)}{d\pi_\sigma}(\gamma)
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Conclusions and generalizations

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\frac{d(\phi^* \mu)}{d\mu}(\gamma) = \exp(-E_{\text{rel}}(\phi^{-1}(\gamma), \gamma)) \frac{d(\phi^* \pi_\sigma)}{d\pi_\sigma}(\gamma)
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\[
E_\Lambda(\phi^{-1}(\gamma)) - E_\Lambda(\gamma)
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Conclusions and generalizations

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\[ E_\Lambda(\phi^{-1}(\gamma)) - E_\Lambda(\gamma) \]

\( \mu \text{ is Diff}_0(X) - \text{ergodic} \iff \mu \in \text{ext}(\mathcal{M}^1_{\text{fm}}(\Gamma) \cap \mathcal{G}_c(V)) \)
Conclusions and generalizations

**Generalizations**

\[
\frac{d(\phi^* \mu)}{d\mu}(\gamma) = \exp(-E_{\text{rel}}(\phi^{-1}(\gamma), \gamma)) \frac{d(\phi^* \pi_\sigma)}{d\pi_\sigma}(\gamma)
\]

\[E_\Lambda(\phi^{-1}(\gamma)) - E_\Lambda(\gamma)\]

- \(\mu\) is Diff\(_0\)(\(X\))–ergodic \(\iff\) \(\mu \in \text{ext}(\mathcal{M}^1_{\text{fm}}(\Gamma) \cap \mathcal{G}_c(V))\)

- The corresponding representation \(V_\mu\) is irreducible
References


• R. S. Ismagilov. Representations of Infinite-Dimensional Groups}, volume 152 of Translations of Mathematical Monographs, 1996