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Tail estimates for positive solutions of stochastic heat equation

HABIB OUERDIANE†¶ and JOSÉ LUIS DA SILVA‡*

†Département de Mathématiques, Faculté des Sciences de Tunis, Université de Tunis El Manar. Campus Universitaire, 1060 Tunis, Tunisie
‡University of Madeira, CCM 9000-390 Funchal, Portugal

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In this paper, we study the solution of a class of stochastic heat equations of convolution type. We give an explicit solution X_t using two basic tools: the characterization theorem for generalized functions and the convolution calculus. For positive initial condition *f* and coefficients processes *Vt*, *Mt*, we prove that the corresponding solution X_t admits an integral representation by a certain measure. Finally, we compute the tail estimate for the obtained solution and its expectation.

Keywords: Generalized functions; Generalized stochastic processes; Integral representation; Convolution product; Heat equation; Tail estimate

AMS Subject Classification: Primary 60H15; Secondary 60H40; 60F10; 46F25; 46G20

1. Introduction

In this work, we consider the following class of the Cauchy problems

$$\begin{cases} \frac{\partial}{\partial t} X_t(\omega, x) = a\Delta X_t(\omega, x) + V_t(\omega, x) * X_t(\omega, x) + M_t(\omega, x) \\ X_0(\omega, x) = f(\omega, x). \end{cases} \tag{1}$$

Here $a \in \mathbb{R}_+$, $t \in [0, \infty)$ is the time parameter, $x \in \mathbb{R}^r$ is the spatial variable, $r = 1, 2, \ldots$ and $\Delta = \sum_{i=1}^r (\partial^2/\partial x_i^2)$ is the Laplacian in the generalized sense on $\mathbb{R}^r$ and ω is the stochastic vector variable in the tempered Schwartz distribution space $S'(\mathbb{R}, \mathbb{R}^d)$, $d \in \mathbb{N}$. The symbol $*$ denotes the usual convolution product between generalized functions. This type of problem was considered by many authors from different point of views, see for example, [5,8,12] and references therein.

In order to study the proposed Cauchy problem, we assume that the initial condition f belongs to a generalized functions space $\mathcal{F}'_\theta(\mathcal{N}')$ (see Section 2 for details and properties) and the coefficients, V_t, M_t are given $\mathcal{F}'_\theta(\mathcal{N}')$-valued generalized processes.

*Corresponding author. Email: luis@uma.pt

¶Email: habib.ouerdiane@fst.rnu.tn

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The paper is organized as follows. In Section 2, we provide the mathematical background needed to solve the Cauchy problem stated above. We construct the appropriate spaces of test $\mathcal{F}_\theta(\mathcal{N}')$ and the associated generalized functions $\mathcal{F}'_\theta(\mathcal{N}')$. Using the Laplace transform we give the characterization theorem for $\mathcal{F}'_\theta(\mathcal{N}')$, cf. Theorem 2.2 below and the basic properties of convolution calculus need later on. In Section 3, we combine the convolution calculus and the characterization theorem in order to find the explicit solution to (1). If we further assume that the coefficients V_t, M_t and the initial condition f are positive distributions, in the sense of Definition 3.5, we show that the solution is associated to a measure which verifies a certain integrability condition, cf. (15). Finally, in Section 4 we use a recent result by Ouerdiane and Privault [9] and apply it to obtain a tail estimate for the positive solution of the Cauchy problem. More precisely, the measure μ_{X_t} which represents the solution verify the inequality

$$\mu_{X_t}(\{u \in \mathcal{M}' | \langle u, \xi\rangle > \alpha\}) \leq C \exp\left(-\beta\left(\frac{\alpha}{m_t|\xi|_{p_t}}\right)\right),$$

where β is a certain Young function, cf. Theorem 4.2. We also compute the generalized (in the sense of Remark 4.5) expectation of the solution X_t.

2. Preliminaries

In this section, we introduce the framework need later on. We start with a real Hilbert space $\mathcal{H} = L^2_d \oplus \mathbb{R}^r$, $L^2_d := L^2(\mathbb{R}, \mathbb{R}^d)$, $d, r = 1, 2, \ldots$ with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. More precisely, if $(f, x) = ((f_1, \ldots, f_d), (x_1, \ldots, x_r)) \in \mathcal{H}$, then the Hilbertian norm of (f, x) is given by:

$$|(f, x)|^2 := \sum_{i=1}^{d} \int_{\mathbb{R}} f_i^2(u)\mathrm{d}u + \sum_{i=1}^{r} x_i^2 = |f|^2_{L^2_d} + |x|^2.$$

We denote by $S_d := S(\mathbb{R}, \mathbb{R}^d)$ the Schwartz test function space and consider the real nuclear triplet

$$\mathcal{M}' = S'_d \oplus \mathbb{R}^r \supset \mathcal{H} \supset S_d \oplus \mathbb{R}^r = \mathcal{M}. \tag{2}$$

The pairing $\langle\cdot,\cdot\rangle$ between $\mathscr{M}'$ and $\mathscr{M}$ is given as an extension of the scalar product in $\mathcal{H}$, i.e. $\langle(\omega, x), (\xi, y)\rangle := (\omega, \xi) + (x, y)$, $(\omega, x) \in \mathcal{H}$ and $(\xi, y) \in \mathcal{M}$. Since $\mathscr{M}$ is a Fréchet nuclear space, then it can be represented as

$$\mathcal{M} = \bigcap_{n=0}^{\infty} S_{d,n} \oplus \mathbb{R}^r = \bigcap_{n=0}^{\infty} \mathcal{M}_n,$$

where $S_{d,n} \oplus \mathbb{R}^r$ is a Hilbert space with norm square given by $|\cdot|_n^2 + |\cdot|^2$, see Ref. [4] and references therein. We will consider the complexification of the triple (2) and denote it by:

$$\mathcal{N}' \supset \mathcal{Z} \supset \mathcal{N}, \tag{3}$$

i.e. $\mathcal{N} = \mathcal{M} + i\mathcal{M}$ and $\mathcal{Z} = \mathcal{H} + i\mathcal{H}$. On $\mathcal{M}'$ we have the standard Gaussian measure γ given by Minlos' theorem via its characteristic functional by

$$C_\mu(\xi, p) = \int_{\mathcal{M}'} \mathrm{e}^{i\langle(\omega,x),(\xi,p)\rangle} \mathrm{d}\mu((\omega, x)) = \exp\left(-\frac{1}{2}(|\xi|^2 + |p|^2)\right), \quad (\xi, p) \in \mathcal{M}.$$

In order to solve the Cauchy problem (1) we need to introduce an appropriate space of generalized functions for which we follow closely the construction in Ref. [6]. Let $\theta = (\theta_1, \theta_2) : \mathbb{R}_+^2 \longrightarrow \mathbb{R}_+, (t_1, t_2) \mapsto \theta_1(t_1) + \theta_2(t_2)$ where θ_1, θ_2 are two Young functions, i.e. θ_i is a continuous, convex, increasing, $\theta_i(0) = 0$ and $\lim_{t\to\infty}(\theta_i(t)/t) = \infty$, $i = 1, 2$. For every pair $m = (m_1, m_2)$ where m_1, m_2 are strictly positive real numbers, we define the Banach space $\mathcal{F}_{\theta,m}(\mathcal{N}_{-n})$, $n \in \mathbb{R}$ by

$$\mathcal{F}_{\theta,m}(\mathcal{N}_{-n}) := \{f : \mathcal{N}_{-n} \longrightarrow \mathbb{C}, \ \text{entire}, \ |f|_{\theta,m,n} < \infty\},$$

where

$$|f|_{\theta,m,n} := \sup_{z\in\mathcal{N}_{-n}} |f(z)| \exp(-\theta(m|z|_{-n}))$$

and for each $z = (\omega, x)$ we have $\theta(m|z|_{-n}) := \theta_1(m_1|\omega|_{-n}) + \theta_2(m_2|x|)$. Here $|\omega|_{-n}$ is the norm in the dual space $S'_{d,n} =: S_{d,-n}$. Now we consider as test function space as the space of entire functions on $\mathcal{N}'$ of (θ_1, θ_2)-exponential growth and minimal type given by

$$\mathcal{F}_\theta(\mathcal{N}') = \bigcap_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}} \mathcal{F}_{\theta,m}(\mathcal{N}_{-n}),$$

endowed with the projective limit topology. Here $\mathbb{R}_+^* :=]0, \infty[$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$. We would like to construct the triplet of the complex Hilbert space $L^2(\mathcal{M}', \mu)$ by $\mathcal{F}_\theta(\mathcal{N}')$. To this end we need another assumption on the pair of Young functions (θ_1, θ_2). Namely, $\lim_{t\to\infty}(\theta_i(t)/t^2) < \infty$, $i = 1, 2$. This is enough to obtain the following Gelfand triplet

$$\mathcal{F}'_\theta(\mathcal{N}') \supset L^2(\mathcal{M}', \mu) \supset \mathcal{F}_\theta(\mathcal{N}'), \tag{4}$$

where $\mathcal{F}'_\theta(\mathcal{N}')$ is the topological dual of $\mathcal{F}_\theta(\mathcal{N}')$ with respect to $L^2(\mathcal{M}', \mu)$. The space $\mathcal{F}'_\theta(\mathcal{N}')$ endowed with the inductive limit topology which coincides with the strong topology since $\mathcal{F}_\theta(\mathcal{N}')$ is a nuclear space, see Ref. [3] for more details on this subject. We denote the duality between $\mathcal{F}'_\theta(\mathcal{N}')$ and $\mathcal{F}_\theta(\mathcal{N}')$ by $\langle\langle \cdot, \cdot \rangle\rangle$ which is the extension of the inner product in $L^2(\mathcal{M}', \gamma)$.

Remark 2.1. For every entire function $f : \mathcal{N}' \longrightarrow \mathbb{C}$ we have the Taylor expansion

$$f(z) = \sum_{k\in\mathbb{N}^2} \langle z^{\otimes k}, f_k \rangle,$$

where $z^{\otimes k} \in \mathcal{N}'^{\hat{\otimes} k}$ and $\hat{\otimes}$ denotes the symmetric tensor product. This allowed us to identify each entire function f with the corresponding Taylor coefficients $\vec{f} = (f_k)_{k\in\mathbb{N}^2}$. The mapping $f \mapsto T(f) = \vec{f}$ is called Taylor series map.

Using the mapping T we can construct a topological isomorphism between the test function space $\mathcal{F}_\theta(\mathcal{N}')$ and the formal power series space $F_\theta(\mathcal{N})$ defined by

$$F_\theta(\mathcal{N}) = \bigcap_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}} F_{\theta,m}(\mathcal{N}_n), \tag{5}$$

where

$$F_{\theta,m}(\mathcal{N}_n) := \left\{ \vec{f} = (f_k)_{k\in\mathbb{N}^2}, f_k \in \mathcal{N}_n^{\hat{\otimes} k} \| \vec{f} \|^2 := \sum_{k\in\mathbb{N}^2} \theta_k^{-2} m^{-k} |f_k|_n^2 < \infty \right\},$$

here $k = (k_1, k_2)$ and $\theta_k^{-2} = \theta_{1,k_1}^{-2}\theta_{2,k_2}^{-2}$ with

$$\theta_{i,k_i} := \inf_{u>0} \frac{\exp(\theta_i(u))}{u^{k_i}}, \quad i = 1, 2.$$

In the case where $\theta(x) = x^2$, then $F_{\theta,1}(\mathcal{N}_n)$ is nothing than the usual Bosonic Fock space associated to $\mathcal{N}_n$, see Ref. [4] for more details.

In applications it is very important to have the characterization of generalized functions $\mathcal{F}'_\theta(\mathcal{N}')$. This is stated in Theorem 2.2 with the help of the Laplace transform. Therefore, let us first define the Laplace transform of an element in $\mathcal{F}'_\theta(\mathcal{N}')$. For every fixed element $(\xi, p) \in \mathcal{N}$ we define the exponential function $\exp((\xi, p))$ by:

$$\mathcal{N}' \ni (\omega, x) \mapsto \exp(\langle \omega, \xi \rangle + (p, x)). \tag{6}$$

It is not hard to verify that for every element $(\xi, p) \in \mathcal{N}$, $\exp((\xi, p)) \in \mathcal{F}_\theta(\mathcal{N}')$. With the help of this function we can define the Laplace transform $\mathcal{L}$ of a generalized function $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ by

$$\hat{\Phi}(\xi, p) := (\mathcal{L}\Phi)(\xi, p) := \langle\langle \Phi, \exp((\xi, p)) \rangle\rangle. \tag{7}$$

The Laplace transform is well defined because $\exp((\xi, p))$ is a test function. In order to obtain the characterization theorem we need to introduce another space of entire functions on $\mathcal{N}$ with θ^*-exponential growth and arbitrary type, where θ^* is another Young function (called polar functions associated to θ) defined by $\theta^*(x_1, x_2) := \theta_1^*(x_1) + \theta_2^*(x_2)$ and

$$\theta_i^*(x_i) := \sup_{t>0}(tx_i - \theta_i(t)), \quad i = 1, 2.$$

The next characterization theorem is essentially based on the topological dual of the formal power series space $F_\theta(\mathcal{N})$ defined in equation (5) and the inverse Taylor series map T^{-1}, see Ref. [2] or [6] for details. In the white noise setting this theorem is known as Potthoff–Streit characterization theorem, see Ref. [7] for details and historical remarks.

THEOREM 2.2. The Laplace transform is a topological isomorphism between $\mathcal{F}'_\theta(\mathcal{N}')$ and the space $\mathcal{G}_{\theta^*}(\mathcal{N})$ which is defined by:

$$\mathcal{G}_{\theta^*}(\mathcal{N}) = \bigcup_{m \in (\mathbb{R}_+^*)^2, n \in \mathbb{N}} \mathcal{G}_{\theta^*,m}(\mathcal{N}_n),$$

and $\mathcal{G}_{\theta^*,m}(\mathcal{N}_n)$ are Banach space of entire functions g on $\mathcal{N}_n$ with the following θ-exponential growth condition

$$|g(\xi, p)| \le k \exp(\theta_1^*(m_1|\xi|_n) + \theta_2^*(m_2|p|)), \quad (\xi, p) \in \mathcal{N}_n,$$

where k, m_1 and m_2 are positive constants.

It is well known that in infinite dimensional complex analysis the convolution operator on a general function space $\mathcal{F}$ is defined as a continuous operator which commutes with the translation operator. This notion generalizes the differential equations with constant coefficients in finite dimensional case. If we consider the space of test functions $\mathcal{F}_\theta(\mathcal{N}')$, then we can show that each convolution operator is associated with a generalized function from $\mathcal{F}'_\theta(\mathcal{N}')$ and vice versa.

Let us define the convolution between a generalized and a test function on $\mathcal{F}'_\theta(\mathcal{N}')$ and $\mathcal{F}_\theta(\mathcal{N}')$, respectively. Let $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ be given, then the convolution $\Phi*\varphi$ is defined by:

$$(\Phi*\varphi)(\omega, x) := \langle\langle \Phi, \tau_{-(\omega,x)}\varphi \rangle\rangle,$$

where $\tau_{-(\omega,x)}$ is the translation operator, i.e.

$$(\tau_{-(\omega,x)}\varphi)(\eta, y) := \varphi(\omega + \eta, x + y).$$

It is not hard to see that $\Phi*\varphi$ is an element of $\mathcal{F}_\theta(\mathcal{N}')$. Note that the dual pairing between $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ is given in terms of the convolution product of Φ and ϕ applied at (0, 0), i.e. $(\Phi*\varphi)(0,0) = \langle\langle \Phi, \varphi \rangle\rangle$.

We can generalize the above convolution product to generalized functions as follows. Let $\Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}')$ be given. Then $\Phi*\Psi$ is defined as

$$\langle\langle \Phi*\Psi, \varphi \rangle\rangle := \langle\langle \Phi, \Psi*\varphi \rangle\rangle, \forall \varphi \in \mathcal{F}_\theta(\mathcal{N}'). \tag{8}$$

This definition of convolution product for generalized functions will be used on Section 3 in order to solve the stochastic heat equation. We have the following connection between the Laplace transform and the convolution product. The simple proof can be found in Ref. [11].

PROPOSITION 2.3. Let $(\xi, p) \in \mathcal{N}$ be given and consider the exponential function $\exp((\xi, p))$ defined on equation (6).

1. Then for every $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ we have

$$\Phi*\exp((\xi, p)) = (\mathcal{L}\Phi)(\xi, p)\exp((\xi, p)).$$

2. For every generalized functions $\Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}')$

$$\mathcal{L}(\Phi*\Psi) = \mathcal{L}\Phi\mathcal{L}\Psi, \tag{9}$$

and equality (9) may be taken as an alternative definition of the convolution product between two generalized functions.

We also need to handle functionals $K : \mathcal{F}'_\theta(\mathcal{N}') \to \mathcal{F}'_\lambda(\mathcal{N}')$ for certain Young functions θ, λ given.

Let $g : \mathbb{C} \to \mathbb{C}$ be an entire function verifying the following growth condition: $|g(z)| \leq C\exp(\gamma(m|z|))$, where $C, m > 0$ and γ is a Young function which not necessary satisfies the condition $\lim_{x\to\infty}(\gamma(x)/x) = \infty$. Then for each $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ the convolution functional $g^*(\Phi)$ defined by:

$$\mathcal{L}(g^*(\Phi)) = g(\mathcal{L}\Phi)$$

belongs to the space $\mathcal{F}'_\lambda(\mathcal{N}')$, where $\lambda = (\gamma \circ e^{\theta^*})^*$, see Ref. [1] for the proof.

In particular if $g(z) = \exp(z)$ and $\gamma(x) = x$, then the convolution exponential

$$\exp^*(\Phi) = \sum_{n=0}^{\infty} \frac{1}{n!}(\Phi^*)^n \tag{10}$$

is a well defined element in $\mathcal{F}'_\lambda(\mathcal{N}')$, where $\lambda = (e^{\theta^*})^*$. The convolution exponential just defined will be the main object in solving the stochastic differential equation in equation (1), cf. (13).

3. Stochastic heat equation of convolution type

A one parameter generalized stochastic process with values in $\mathcal{F}'_\theta(\mathcal{N}')$ is a family of distributions $\{\Phi_t, t \in I\} \subset \mathcal{F}'_\theta(\mathcal{N}')$, where I is an interval from $\mathbb{R}$. Without loss generality we may assume that $0 \in I$. The process Φ_t is said to be continuous if the map $t \mapsto \Phi_t$ is continuous. In order to introduce generalized stochastic integrals, we need the following result proved in Ref. [10].

Proposition 3.1. Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence of generalized functions in $\mathcal{F}'_\theta(\mathcal{N}')$. Then the following two conditions are equivalent:

1. The sequence $(\Phi_n)_{n\in\mathbb{N}}$ converges in $\mathcal{F}'_\theta(\mathcal{N}')$ strongly.
2. The sequence $(\hat{\Phi}_n = \mathcal{L}(\Phi_n))_{n\in\mathbb{N}}$ of Laplace transform of $(\Phi_n)_{n\in\mathbb{N}}$ satisfies the following two conditions:
   (a) There exists $p \in \mathbb{N}$ and $m \in \left(\mathbb{R}_+^*\right)^2$ such that the sequence $(\hat{\Phi}_n)_{n\in\mathbb{N}}$ belongs to $\mathcal{G}_{\theta^*,m}(\mathcal{N}_p)$ and is bounded in this Banach space.
   (b) For every point $z \in \mathcal{N}$, the sequence of complex numbers $(\hat{\Phi}_n(z))_{n=0}^{\infty}$ converges.

Let $\{\Phi_t\}_{t\in I}$ be a continuous $\mathcal{F}'_\theta(\mathcal{N}')$-process and put

$$\Phi_n = \frac{t}{n}\sum_{k=0}^{n-1}\Phi_{(tk/n)}, \quad n \in \mathbb{N}^* := \mathbb{N}\backslash\{0\}, \quad t \in I.$$

It is easy to prove that the sequence $(\hat{\Phi}_n)$ is bounded in $\mathcal{G}_{\theta^*}(\mathcal{N})$ and for every $\xi \in \mathcal{N}, p \in \mathbb{C}^r$ $(\hat{\Phi}_n(\xi, p))_n$ converges to $\int_0^t \hat{\Phi}_s(\xi, p)\mathrm{d}s$. Thus, we conclude by Proposition 3.1 that (Φ_n) converges in $\mathcal{F}'_\theta(\mathcal{N}')$. We denote its limit by

$$\int_0^t \Phi_s \mathrm{d}s := \lim_{n\to\infty}\Phi_n \quad \text{in } \mathcal{F}'_\theta(\mathcal{N}'). \tag{11}$$

The result of the following proposition is widely used in this remaining of this paper, the proof is given in Ref. [11].

Proposition 3.2. For a given continuous generalized stochastic process X_t we define the generalized function

$$Y_t(x,\omega) = \int_0^t X_s(x,\omega)\mathrm{d}s \in \mathcal{F}'_\theta(\mathcal{N}')$$

by

$$\mathcal{L}\left(\int_0^t X_s(x,\omega)\mathrm{d}s\right)(\xi,p) := \int_0^t \mathcal{L}X_s(p,\xi)\mathrm{d}s.$$

Moreover, the generalized stochastic process $Y_t(x,\omega)$ is differentiable in $\mathcal{F}'_\theta(\mathcal{N}')$ and we have $(\partial/\partial t)Y_t(x,\omega) = X_t(x,\omega)$.

We are now ready to solve the Cauchy problem in equation (1). Let us recall again this problem for the reader convenience. Let f be a given generalized function in $\mathcal{F}'_{\theta}(\mathcal{N}')$ and V_t, M_t given $\mathcal{F}'_{\theta}(\mathcal{N}')$-valued continuous generalized stochastic processes. Consider the following stochastic differential equation with initial condition f and coefficients V_t, M_t

$$\begin{cases} \frac{\partial}{\partial t}X_t(\omega,x) = a\Delta X_t(\omega,x) + V_t(\omega,x)*X_t(\omega,x) + M_t(\omega,x) \\ X_0(\omega,x) = f(\omega,x), \end{cases} \tag{12}$$

where a is a positive constant and Δ is the Laplacian in the generalized sense with respect to the spacial variable $x \in \mathbb{R}^r$.

Theorem 3.3. The Cauchy problem (12) has a unique solution X_t which is a generalized $\mathcal{F}^*_{\beta}(\mathcal{N}')$-valued stochastic process, where the Young function β is given by $\beta = (\mathrm{e}^{\theta^*})^*$. Moreover, the solution X_t is given explicitly by

$$\begin{aligned} X_t(\omega,x) &= f(\omega,x)*\exp^*\left(\int_0^t V_s(\omega,x)\mathrm{d}s\right)*\gamma_{2at} \\ &\quad + \int_0^t \exp^*\left(\int_s^t V_u(\omega,x)\mathrm{d}u\right)*\gamma_{2a(t-s)}*M_s\mathrm{d}s. \end{aligned} \tag{13}$$

where γ_{2at} is the centered Gaussian measure on $\mathbb{R}^r$ with variance $2at$.

Proof. To obtain the solution (13) at first we apply the Laplace transform to equation (12) which reduces the problem to a ordinary differential equation. Then the result follows by the characterization Theorem 2.2. □

Remark 3.4. For $a = 0$ the Cauchy problem (12) reduces to

$$\begin{cases} \frac{\partial}{\partial t}X_t(\omega,x) = V_t(\omega,x)*X_t(\omega,x) + M_t(\omega,x) \\ X_0(\omega,x) = f(\omega,x). \end{cases} \tag{14}$$

Taking into account that $\gamma_{2at} \to \delta_0$, $a \to 0$, where δ_0 denotes the Dirac measure at 0 which is the unit element for the convolution product, then the solution (13) reduces to

$$X_t = f(\omega,x)*\exp^*\left(\int_0^t V_s(\omega,x)\mathrm{d}s\right) + \int_0^t \exp^*\left(\int_s^t V_u(\omega,x)\mathrm{d}u\right)*M_s\mathrm{d}s.$$

The problem (14) was studied in other works, see for example, Ref. [1]. Our solution coincides with their solution.

In the next section, we also need the notion of positive distributions. Therefore, we recall this notion and the connection between positive distributions and measures as well its characterization.

Definition 3.5. 1. Let $\mathcal{F}_{\theta}(\mathcal{N}')_+$ denote the cone of positive test functions, i.e. $\varphi \in \mathcal{F}_{\theta}(\mathcal{N}')_+$ if $\varphi(u + i0) \geq 0$ for all $u \in \mathcal{M}'$.

2. The space $\mathcal{F}'_\theta(\mathcal{N}')_+$ of positive distributions is a subset of $\Phi \in {}'\mathcal{F}_\theta(\mathcal{N}')$ such that $\langle\langle \Phi, \varphi \rangle\rangle \geq 0$, for all $\varphi \in \mathcal{F}_\theta(\mathcal{N}')_+$.

The following theorem gives the integral representation for positive distributions as measures and their characterization. For details we refer to Ref. [10] and references therein.

THEOREM 3.6. Let $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')_+$ be a given positive distribution. Then there exists a unique Radon measure μ_Φ on $\mathcal{M}'$ such that

$$\langle\langle \Phi, \varphi \rangle\rangle = \int_{\mathcal{M}'} \varphi(u + i0) \mathrm{d}\mu_\Phi(u), \quad \varphi \in \mathcal{F}_\theta(\mathcal{N}').$$

Conversely, for each finite positive Borel measure μ on $\mathcal{M}'$, μ represents a positive distribution in $\mathcal{F}'_\theta(\mathcal{N}')_+$ if and only if there exists $p, m > 0$ such that μ is supported by $\mathcal{M}_{-p}$ and

$$\int_{\mathcal{M}_{-p}} \mathrm{e}^{\theta(m|u|_{-p})} \mathrm{d}\mu(u) < \infty.$$

LEMMA 3.7. Let $\Phi_1, \Phi_2 \in \mathcal{F}'_\theta(\mathcal{N}')_+$ be positive distributions. Then $\Phi_1 * \Phi_2$ and $\mathrm{e}^{*\Phi_1}$ are positive distributions.

Proof. Using equality (8) it is sufficient to show that the convolution product between a generalized function and a positive test function is a positive test function. In fact, if $\varphi \in \mathcal{F}_\theta(\mathcal{N}')_+$ then

$$(\Phi_2 * \varphi)(u + i0) := \langle \Phi_2, \tau_{-u}\varphi \rangle, \quad u \in \mathcal{M}'$$

and the result follows since $(\tau_{-u}\varphi)(v + i0) := \varphi(u + v) \geq 0$, for all $u, v \in \mathcal{M}'$. As a consequence we have $\Phi_1^{*n} \in \mathcal{F}'_\theta(\mathcal{N}')_+$, $n \in \mathbb{N}$. Now we use equality (10) to derive the positivity of $\mathrm{e}^{*\Phi_1}$. $\square$

As a corollary of this lemma we give sufficient conditions on f, V_t and M_t such that the solution (13) of the Cauchy problem (12) is a positive generalized function.

COROLLARY 3.8. Suppose that $f, V_t, M_t \in \mathcal{F}'_\theta(\mathcal{N}')_+$ for any $t \in [0, \infty)$. Then the solution (13) is a positive distribution and thus there exists a unique Radon measure μ_{X_t} associated to X_t, i.e.

$$\langle\langle X_t, \varphi \rangle\rangle = \int_{\mathcal{M}'} \varphi(u) \mathrm{d}\mu_{X_t}(u), \quad \varphi \in \mathcal{F}_\beta(\mathcal{N}').$$

Moreover, there exist $m, p > 0$ such that μ_{X_t} satisfies the integrability condition

$$\int_{\mathcal{M}_{-p}} \mathrm{e}^{\beta(m|u|_{-p})} \mathrm{d}\mu_{X_t}(u) < \infty, \quad \beta = (\mathrm{e}^{\theta^*})^*. \tag{15}$$

Proof. First we notice that $\int_0^t V_s(\omega, x)\mathrm{d}s$ is a positive distribution which follows directly from the definition (11) and Proposition 3.1. The result follows using the associativity of the convolution product, the previous Lemma and Theorem 3.6. □

4. Tail estimates and expectation of the solution

In this section, we will use the previous results in order to obtain the tail estimate for the solution X_t represented by the measure μ_{X_t} in Corollary 3.8. We also compute the generalized expectation of X_t, cf. Theorem 4.4.

At first we state an independent result for positive generalized functions, see Theorem 2.1 in Ref. [9].

THEOREM 4.1. Let $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')_+$ be a given positive distribution and μ_Φ the associated measure. Consider for any $\xi \in \mathcal{M}$, $\alpha \in \mathbb{R}$ the half-plane $A_{\xi,\alpha}$ in $\mathcal{M}'$ defined by

$$A_{\xi,\alpha} := \{u \in \mathcal{M}' | \langle u, \xi\rangle > \alpha\}.$$

Then there exists constants $m > 0$, $p \in \mathbb{N}$ such that

$$\mu_\Phi(A_{\xi,\alpha}) \leq C \exp\left(-\theta\left(\frac{\alpha}{m|\xi|_p}\right)\right), \tag{16}$$

where $C = |\hat{\Phi}|_{\theta,m,p}$.

THEOREM 4.2. Suppose that $f, V_t, M_t \in \mathcal{F}'_\theta(\mathcal{N}')_+$ for any $t \in [0, \infty)$. Then there exits a unique positive Radon measure μ_{X_t} on $\mathcal{M}'$ associated to the solution X_t of the Cauchy problem (12) given in equation (13) such that

$$\mu_{X_t}(A_{\xi,\alpha}) \leq C_t \exp\left(-\beta\left(\frac{\alpha}{m_t|\xi|_{p_t}}\right)\right), \tag{17}$$

where $\beta = (\mathrm{e}^{\theta^*})^*$ and certain $C_t, m_t, p_t > 0$, $t \in [0, \infty)$.

Proof. It is clear that the solution X_t in equation (13) belongs to $\mathcal{F}'_\beta(\mathcal{N}')_+$ using Lemma 3.7. The existence and uniqueness of the Radon measure μ_{X_t} on $\mathcal{M}'$ associated to X_t follows from Theorem 3.6. Finally, the estimate (17) is a consequence of the inequality (16) with θ replaced by β. □

LEMMA 4.3. Let $\Phi_1, \Phi_2 \in \mathcal{F}'_\theta(\mathcal{N}')$ be given and $1 \in \mathcal{F}_\theta(\mathcal{N}')$ the constant test function identically equal to 1. Then we have the following equalities

$$\langle\!\langle \Phi_1 * \Phi_2, 1\rangle\!\rangle = \langle\!\langle \Phi_1, 1\rangle\!\rangle \langle\!\langle \Phi_2, 1\rangle\!\rangle, \quad \langle\!\langle \mathrm{e}^{*\Phi_1}, 1\rangle\!\rangle = \mathrm{e}^{\langle\!\langle \Phi_1, 1\rangle\!\rangle}.$$

Proof. In fact, we have $\langle\!\langle \Phi_1 * \Phi_2, 1\rangle\!\rangle := \langle\!\langle \Phi_1, \Phi_2 * 1\rangle\!\rangle$ and we notice that

$$(\Phi_2 * 1)(u) := \langle\!\langle \Phi_2, \tau_{-u} 1\rangle\!\rangle = \langle\!\langle \Phi_2, 1\rangle\!\rangle.$$

It follows from this equality that $\langle \Phi_1^{*n}, 1\rangle = \langle\!\langle \Phi_1, 1\rangle\!\rangle^n$. The second equality of the Lemma is a consequence of equation (10). $\square$

THEOREM 4.4. The solution of the Cauchy problem X_t in equation (13) satisfies the following equality:

$$\langle\!\langle X_t, 1\rangle\!\rangle = \langle\!\langle f, +1\rangle\!\rangle \exp\left(\int_0^t \langle\!\langle V_s, 1\rangle\!\rangle \mathrm{d}s\right) + \int_0^t \exp\left(\int_s^t \langle\!\langle V_u, 1\rangle\!\rangle \mathrm{d}u\right)\langle\!\langle M_s, 1\rangle\!\rangle \mathrm{d}s.$$

Proof. The equality is a consequence of the previous Lemma, the associativity of the convolution product and the fact that $\langle \gamma_{2at}, 1\rangle = 1$. $\square$

Remark 4.5. The bilinear dual pairing $\langle\!\langle X_t, 1\rangle\!\rangle$ may be interpreted as a generalized expectation of X_t, denoted by $\mathbb{E}_\mu(X_t)$, in connection with the triple (4). In fact, if $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ is a random variable on the probability space $(\mathcal{M}', \mathcal{B}(\mathcal{M}'), \mu)$, then its expectation is given by:

$$\mathbb{E}_\mu(\varphi) = \int_{\mathcal{M}'} \varphi(u) d\mu(u) = ((\varphi, 1))_{L^2(\mathcal{M}',\mu)}.$$

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