A nonlinear stochastic equation of convolution type: Solution and stochastic representation

F. Cipriano[*], H. Ouerdiane[†], J. L. Silva[‡] and R. Vilela Mendes[§]

Abstract

A nonlinear equation, obtained from the Burgers' equation by replacing the usual product by a convolution product, is studied. The initial condition is a generalized stochastic process. Using the Laplace transform in a general white noise analysis setting, a general solution is found in $(1+n)$-dimensions and a stochastic representation for the Laplacian transform of the solution is obtained.

Keywords: Nonlinear Stochastic equation; Convolution calculus; Laplace transform; Generalized functions.
Mathematics Subject Classification 2000: 60H15, 60H40, 46F25.

1 Introduction

The aim of this paper is to study the following nonlinear stochastic equation of convolution type

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} + (\vec{u} * \vec{\nabla})\vec{u} = \nu \Delta \vec{u} + \vec{f} * \vec{u} \\ \\ \vec{u}(0,x) = \vec{u}_0(x), \end{cases} \tag{1}$$

[*]GFM e Dep. de Matemática FCT-UNL, Av. Prof. Gama Pinto 2, 1649-003, Lisboa, Portugal, cipriano@gfm.cii.fc.ul.pt

[†]Département de Mathématiques, Faculté des Sciences de Tunis, 1060 Tunis, Tunisia, habib.ouerdiane@fst.rnu.tn

[‡]University of Madeira, CCM 9000-390 Funchal, Portugal, luis@uma.pt

[§]CMAF, Complexo Interdisciplinar, Universidade de Lisboa, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal, vilela@cii.fc.ul.pt

where $\vec{u}$ is a $\mathbb{R}^n$-generalized vector field, $\vec{f}$ is a n-dimensional generalized function, $\nu > 0$ a real constant, $t \in [0, \infty)$ the time parameter, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the spatial variable, Δ the Laplacian operator in $\mathbb{R}^n$, ∇ the gradient and $*$ the convolution product for generalized functions (see [1], [11] and Subsection 2.2 for more details), $\vec{f} * \vec{u}$ is a driving term, $\vec{u} * \vec{\nabla}$ denotes the differential operator

$$\sum_{j=1}^{n} u_j * \frac{\partial}{\partial x_j}$$

and the initial condition $\vec{u}_0 = (u_{0,1}, \cdots, u_{0,n})$ is a n-dimensional generalized stochastic process, see Section 3 for more details.

Problem (1) with $*$ replaced by the usual product would coincide with the Burgers equation well known in the literature ([3], [5] and references therein). However the physical interpretation of (1) is quite different. This is easily seen by comparing the j-component of the Fourier transform of the nonlinear kinetic term $(\vec{u}.\nabla)\vec{u}$ in the Burgers equation, namely

$$\int_{\mathbb{R}^n} d^n q \sum_{i=1}^{n} \tilde{u}_i(t, q)(k - q)_i \tilde{u}_j(t, k - q) \tag{2}$$

with the j-component of the Fourier transform of the nonlinear term in (1), $(\vec{u} * \nabla)\vec{u}$

$$\sum_{i=1}^{n} \tilde{u}_i(t, k) k_i \tilde{u}_j(t, k) \tag{3}$$

$\tilde{u}_i$ denotes the Fourier transform of u_i.

In the Burgers case the expression (2) implies that the Fourier modes at length scale $\frac{2\pi}{q}$ and $\frac{2\pi}{k-q}$ control the eddies at scale $\frac{2\pi}{k}$, consistent with the phenomenological description of the inertial range in the turbulence cascade. However, in the convolution case the nonlocal nonlinearity corresponds to a self-interaction of the modes at each length scale. Therefore the convolution equation has a different physical interpretation. Nevertheless, nonlocal nonlinearities are also important in models of transport in magnetized plasmas, see [4] and also in the modeling of convection driven by density gradients as it arises in geophysical fluid flows, see [12], [13], [14] and [15].

By restricting oneself to solutions of gradient type, the Burgers equation may be linearized by the Cole-Hopf transformation. This provides the most general solution for (1+1)-dimensions but not for $(1+n)$-dimensions. In contrast, for our equation (1), using the Laplace transform in a general setting, we obtain a general solution for $(1+n)$-dimensions.

The paper is organized as follows: In Section 2 we provide the mathematical background needed to solve the Cauchy problem stated above, namely the spaces of test and generalized functions, the characterization theorem of generalized functions and the convolution product as well as some of its properties. In Section 3 we combine the convolution calculus and the characterization theorem in order to find an explicit solution of the problem (1).

Finally, in Section 4 we obtain a stochastic representation of the Laplace transform of the solution. Stochastic representations of differential equations and their solutions, not only provide a new interpretation of the solutions but are also useful for existence results and in the study of the fluctuations associated to the phenomena for which the equation represents a mean-field approximation. For a striking recent example of such use of stochastic representations we refer to the work done on the Navier-Stokes equation (see [16] and references therein).

2 Preliminaries

2.1 Test and generalized functions spaces

In this section we introduce the framework needed later on. The starting point is the real Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r$, $d, r \in \mathbb{N}$ with scalar product $(\cdot, \cdot)$ and norm $|\cdot|$. More precisely, if $(f, x) = ((f_1, \ldots, f_d), (x_1, \ldots, x_r)) \in \mathcal{H}$, then

$$|(f,x)|^2 := \sum_{i=1}^{d} \int_{\mathbb{R}} f_i^2(u)du + \sum_{i=1}^{r} x_i^2 = |f|_{L^2}^2 + |x|_{\mathbb{R}^r}^2.$$

Let us consider the real nuclear triplet

$$\mathcal{M}' = S'(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r \supset \mathcal{H} \supset S(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r = \mathcal{M}. \tag{4}$$

The pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{M}'$ and $\mathcal{M}$ is given in terms of the scalar product in $\mathcal{H}$, i.e., $\langle (\omega, x), (\xi, p) \rangle := (\omega, \xi)_{L^2} + (x, p)_{\mathbb{R}^r}$, $(\omega, x) \in \mathcal{M}'$ and $(\xi, p) \in \mathcal{M}$. Since $\mathcal{M}$ is a Fréchet nuclear space, it can be represented as

$$\mathcal{M} = \bigcap_{n=0}^{\infty} S_n(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r = \bigcap_{n=0}^{\infty} \mathcal{M}_n,$$

where $S_n(\mathbb{R},\mathbb{R}^d)\times\mathbb{R}^r$ is a Hilbert space with norm squared given by $|\cdot|_n^2+|\cdot|_{\mathbb{R}^r}^2$, see e.g. [8] or [2] and references therein. We will consider the complexification of the triple (4) and denote it by

$$\mathcal{N}' \supset \mathcal{Z} \supset \mathcal{N}, \tag{5}$$

where $\mathcal{N}=\mathcal{M}+i\mathcal{M}$ and $\mathcal{Z}=\mathcal{H}+i\mathcal{H}$. On $\mathcal{M}'$ we have the standard Gaussian measure γ given by Minlos's theorem via its characteristic functional, namely for every $(\xi,p)\in\mathcal{M}$

$$C_\gamma(\xi,p)=\int_{\mathcal{M}'}\exp(i\langle(\omega,x),(\xi,p)\rangle)d\gamma((\omega,x))=\exp\left(-\frac{1}{2}(|\xi|_{L^2}^2+|p|_{\mathbb{R}^r}^2)\right).$$

In order to solve the $(1+n)$-dimensional equation of convolution type (1) we need to introduce an appropriate space of vectorial generalized functions. We borrow this construction from [9]. Let $\theta=(\theta_1,\theta_2):\mathbb{R}_+^2\to\mathbb{R}$, $(t_1,t_2)\mapsto\theta_1(t_1)+\theta_2(t_2)$ where θ_1,θ_2 are two Young functions, i.e., $\theta_i:\mathbb{R}_+\to\mathbb{R}_+$ continuous convex strictly increasing function and

$$\lim_{t\to\infty}\frac{\theta_i(t)}{t}=\infty,\quad \theta_i(0)=0,\quad i=1,2.$$

For every pair $m=(m_1,m_2)$ with $m_1,m_2\in]0,\infty[$, we define the Banach space $\mathcal{F}_{\theta,m}(\mathcal{N}_{-n})$, $n\in\mathbb{N}$ by

$$\mathcal{F}_{\theta,m}(\mathcal{N}_{-n}):=\{f:\mathcal{N}_{-n}\to\mathbb{C},\ \text{entire},\|f\|_{\theta,m,n}<\infty\},$$

where

$$\|f\|_{\theta,m,n}:=\sup_{z\in\mathcal{N}_{-n}}\{|f(z)|\exp(-\theta(m|z|_{-n}),$$

Here, for each $z=(\omega,x)$ we have $\theta(m|z|_{-n}):=\theta_1(m_1|\omega|_{-n})+\theta_2(m_2|x|)$. Now we consider as test function space the space of entire functions on $\mathcal{N}'$ of (θ_1,θ_2)-exponential growth and minimal type

$$\mathcal{F}_\theta(\mathcal{N}'):=\bigcap_{m\in(\mathbb{R}_+^*)^2,n\in\mathbb{N}_0}\mathcal{F}_{\theta,m}(\mathcal{N}_{-n}),$$

endowed with the projective limit topology. We would like to use $\mathcal{F}_\theta(\mathcal{N}')$ to construct a triple centered in the complex Hilbert space $L^2(\mathcal{M}',\gamma)$. To this end we need another condition on the pair of Young functions (θ_1, θ_2). Namely,

$$\lim_{t\to\infty} \frac{\theta_i(t)}{t^2} < \infty, \quad i = 1, 2. \tag{6}$$

This is enough to obtain the following Gelfand triple

$$\mathcal{F}'_\theta(\mathcal{N}') \supset L^2(\mathcal{M}', \gamma) \supset \mathcal{F}_\theta(\mathcal{N}'), \tag{7}$$

where $\mathcal{F}'_\theta(\mathcal{N}')$ is the topological dual of $\mathcal{F}_\theta(\mathcal{N}')$ with respect to $L^2(\mathcal{M}', \gamma)$ endowed with the inductive limit topology.

In applications it is very important to have the characterization of generalized functions from $\mathcal{F}'_\theta(\mathcal{N}')$. First we define the Laplace transform of an element in $\mathcal{F}'_\theta(\mathcal{N}')$. For every fixed element $(\xi, p) \in \mathcal{N}$ the exponential function $\exp((\xi, p))$ is a well defined element in $\mathcal{F}_\theta(\mathcal{N}')$, see [7]. The Laplace transform $\mathcal{L}$ of a generalized function $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ is defined by

$$\hat{\Phi}(\xi, p) := (\mathcal{L}\Phi)(\xi, p) := \langle\!\langle \Phi, \exp((\xi, p)) \rangle\!\rangle. \tag{8}$$

We are ready to state to characterization theorem (see e.g., [7] and [1] for the proof) which is the main tool in our further consideration.

Theorem 2.1 *1. The Laplace transform is a topological isomorphism between $\mathcal{F}'_\theta(\mathcal{N}')$ and the space $\mathcal{G}_{\theta^*}(\mathcal{N})$, where $\mathcal{G}_{\theta^*}(\mathcal{N})$ is defined by*

$$\mathcal{G}_{\theta^*}(\mathcal{N}) := \bigcup_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0} \mathcal{G}_{\theta^*,m}(\mathcal{N}_n),$$

and $\mathcal{G}_{\theta^*,m}(\mathcal{N}_n)$ is the space of entire functions on $\mathcal{N}_n$ with the following θ-exponential growth condition

$$\mathcal{G}_{\theta^*,m}(\mathcal{N}_n) \ni g, \; |g(\xi, p)| \le k \exp(\theta_1^*(m_1|\xi|_n) + \theta_2^*(m_2|p|)), \; (\xi, p) \in \mathcal{N}_n.$$

Here $\theta^* = (\theta_1^*, \theta_2^*)$, where $\theta_i^* = \sup_{t\ge 0}(tx - \theta_i(t))$ is the Legendre transform associated to the function θ_i, $i = 1, 2$.

2. In the particular case $\theta(x) = (\theta_1(x), \theta_2(x)) = (x, x)$, we denote the space $\mathcal{F}'_\theta(\mathcal{N}')$ by $\mathcal{F}'_x(\mathcal{N}')$. Then the Laplace transform realizes a topological isomorphism between the distributions space $\mathcal{F}'_x(\mathcal{N}')$ and the space $Hol_0(\mathcal{N})$ of holomorphic function on a neighborhood of zero of $\mathcal{N}$.

2.2 The Convolution Product $*$

It is well known that in infinite dimensional complex analysis the convolution operator on a general function space $\mathcal{F}$ is defined as a continuous operator which commutes with the translation operator. Let us define the convolution between a generalized and a test function. Let $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ be given, then the convolution $\Phi * \varphi$ is defined by

$$(\Phi * \varphi)(\omega, x) := \langle\langle \Phi, \tau_{-(\omega,x)}\varphi \rangle\rangle,$$

where $\tau_{-(\omega,x)}$ is the translation operator, i.e.,

$$(\tau_{-(\omega,x)}\varphi)(\eta, y) := \varphi(\omega + \eta, x + y).$$

It is not hard the see that $\Phi * \varphi \in \mathcal{F}_\theta(\mathcal{N}')$, cf. [7]. The convolution product is given in terms of the dual pairing as $(\Phi * \varphi)(0, 0) = \langle\langle \Phi, \varphi \rangle\rangle$ for any $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$.

We can generalize the above convolution product for generalized functions as follows. Let $\Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}')$ be given, then $\Phi * \Psi$ is defined by

$$\langle\langle \Phi * \Psi, \varphi \rangle\rangle := \langle\langle \Phi, \Psi * \varphi \rangle\rangle, \ \forall \varphi \in \mathcal{F}_\theta(\mathcal{N}'). \tag{9}$$

This definition of convolution product for generalized functions will be used later for the solution of the equation (1). We have the following equality, (see [11], Proposition 3):

$$\Phi * \exp((\xi, p)) = (\mathcal{L}\Phi)(\xi, p) \exp((\xi, p)), \ (\xi, p) \in \mathcal{N}.$$

As a consequence of the above equality and definition (9) we obtain

$$\mathcal{L}(\Phi * \Psi) = \mathcal{L}\Phi\mathcal{L}\Psi, \ \Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}') \tag{10}$$

which says that the Laplace transform maps the convolution product on $\mathcal{F}'_\theta(\mathcal{N}')$ into the usual pointwise product in the algebra of functions $\mathcal{G}_{\theta^*}(\mathcal{N})$. Therefore we may use Theorem 2.1 to define the convolution product between two generalized functions as

$$\Phi * \Psi = \mathcal{L}^{-1}(\mathcal{L}\Phi\mathcal{L}\Psi).$$

This allows us to introduce the convolution exponential of a generalized function. In fact, for every $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ we may easily check that $\exp(\mathcal{L}\Phi) \in$

$\mathcal{G}_{e^{\theta^*}}(\mathcal{N})$. Using the inverse Laplace transform and the fact that any Young function θ verifies the property $(\theta^*)^* = \theta$ we obtain that $\mathcal{L}^{-1}(\mathcal{G}_{e^{\theta^*}}(\mathcal{N})) = \mathcal{F}'_{(e^{\theta^*})^*}(\mathcal{N}')$. Now we give the definition of the convolution exponential of $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$, denoted by $\exp^* \Phi$

$$\exp^* \Phi := \mathcal{L}^{-1}(\exp(\mathcal{L}\Phi)).$$

Notice that $\exp^* \Phi$ is a well defined element in $\mathcal{F}'_{(e^{\theta^*})^*}(\mathcal{N}')$ and therefore the distribution $\exp^* \Phi$ is given in terms of a convergent series

$$\exp^* \Phi = \delta_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \Phi^{*n}, \tag{11}$$

where Φ^{*n} is the convolution of Φ with itself n times, $\Phi^{*0} := \delta_0$ by convention with δ_0 denoting the Dirac distribution at 0. We refer to [1] for more details concerning convolution product on $\mathcal{F}'_\theta(\mathcal{N}')$.

A one parameter generalized stochastic process with values in $\mathcal{F}'_\theta(\mathcal{N}')$ is a family of generalized functions $\{\Phi(t),\ t \geq 0\} \subset \mathcal{F}'_\theta(\mathcal{N}')$. The process $\Phi(t)$ is said to be continuous if the map $t \mapsto \Phi(t)$ is continuous. For a given continuous generalized stochastic process $(X(t))_{t\geq 0}$ we define the generalized stochastic process

$$Y(t,\omega,x) = \int_0^t X(s,\omega,x)ds \in \mathcal{F}'_\theta(\mathcal{N}')$$

by

$$\mathcal{L}\left(\int_0^t X(s,\omega,x)ds\right)(\xi,p) := \int_0^t \mathcal{L}X(s,\xi,p)ds. \tag{12}$$

The process $Y(t,\omega,x)$ is differentiable and we have $\frac{\partial}{\partial t}Y(t,\omega,x) = X(t,\omega,x)$. The details of the proof can be seen in [10], Proposition 11.

2.3 Convolution inverse of distributions

Let Φ a fixed element on the distribution space $\mathcal{F}'_\theta(\mathcal{N}')$ and consider the following convolution equation

$$\Phi * \Psi = \delta_0, \tag{13}$$

Applying the Laplace transform to the convolution equation (13) we obtain

$$\hat{\Phi}.\hat{\Psi} = 1.$$

If $\hat{\Phi}(\xi, q) \neq 0$ for every $(\xi, q) \in \mathcal{N}$, then using the division result in the space $\mathcal{G}_{\theta^*}(\mathcal{N})$ (see [6]) we obtain

$$\hat{\Psi} = \frac{1}{\hat{\Phi}} \in \mathcal{G}_{\theta^*}(\mathcal{N}).$$

Moreover, by the Laplace transform isomorphism (see Theorem 2.1), we prove the existence and uniqueness of the solution $\Psi \in \mathcal{F}'_\theta(\mathcal{N}')$ in the equation (13). If we denote this solution Ψ by Φ^{*-1}, we have

$$\mathcal{L}\big(\Phi^{*-1}\big) = \frac{1}{\mathcal{L}(\Phi)}.$$

This division result is also true in the limit case $\theta(x) = (x, x)$; i. e., $\hat{\Phi} \in \mathcal{G}_{\theta^*}(\mathcal{N}) = Hol_0(\mathcal{N})$.

3 Solution of the n-dimensional convolution equation

We are now ready to solve the Cauchy problem stated in (1) which we recall for the reader convenience, namely

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} + (\vec{u} * \vec{\nabla})\vec{u} = \nu \Delta \vec{u} + \vec{f} * \vec{u} \\ \\ \vec{u}(0, x) = \vec{u}_0(x). \end{cases} \tag{14}$$

The different terms in (14) are as follows: $\vec{u}_0(x) = (u_{0,1}(x), \ldots, u_{0,n}(x))$ is a generalized function; $u_{0,j}(x) \in \mathcal{F}'_\theta(\mathcal{N}')$, $\nu > 0$ a real constant, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, $\vec{u} = (u_1, \ldots, u_n)$, $\vec{f} = (f_1, \ldots, f_n)$, with $f_j = f_j(t, x), u_j$

$\in \mathcal{F}'_\theta(\mathcal{N}')$. The three terms $\vec{f} * \vec{u}$, $\Delta\vec{u}$ and $(\vec{u} * \nabla)\vec{u}$ corresponds to

$$\begin{aligned}
\vec{f} * \vec{u} &= (f_1 * u_1, \ldots, f_n * u_n) \\
\Delta\vec{u} &= \begin{pmatrix} \Delta u_1 \\ \vdots \\ \Delta u_n \end{pmatrix} \\
(\vec{u} * \nabla)\vec{u} &= \begin{pmatrix} \displaystyle\sum_{j=1}^{n} u_j * \partial_j u_1 \\ \vdots \\ \displaystyle\sum_{j=1}^{n} u_j * \partial_j u_n \end{pmatrix}, \quad \partial_j := \frac{\partial}{\partial x_j},\ j = 1, \ldots, n.
\end{aligned}$$

We are now ready to prove the main result of this paper, namely, using the tools from Section 2, we obtain an explicit solution of the equation (14).

Theorem 3.1 *Let $\vec{u}_0(x) = (u_{0,1}(x), \ldots, u_{0,n}(x))$ and $\vec{f} = (f_1, \ldots, f_n)$ be such that $u_{0,k}, f_k \in \mathcal{F}'_x(\mathcal{N}')$, $k = 0, \ldots, n$. Then the solution $\vec{u}(t, \omega, x)$ of the nonlinear equation of convolution type (14) is given explicitly by the following system:*

$$\begin{aligned}
u_k(t, x) &= u_{0,k}(x) * e^{* \int_0^t f_k(s) ds} * \gamma_{2\nu t} \\
&\quad * \left(\delta_0 + \sum_{j=1}^{n} \partial_j u_{0,j} * \int_0^t e^{* \int_0^\tau f_j(s) ds} * \gamma_{2\nu\tau}\, d\tau \right)^{*-1}
\end{aligned} \tag{15}$$

with $k = 1, \ldots, n$ and δ_0 is the Dirac measure at point zero and $\gamma_{2\nu t}$ is the Gaussian measure with variance $2\nu t$ on $\mathbb{R}^n$.

Proof. We denote $\partial_t = \frac{\partial}{\partial t}$ such that the n-dimensional equation (14) may be written as

$$\begin{cases} \partial_t u_k + \displaystyle\sum_{j=1}^{n} u_j * \partial_j u_k = \nu \Delta u_k + f_k * u_k \\ u_k(0, \omega, x) = u_{0,k}(\omega, x) \end{cases} \tag{16}$$

where $k = 1, \ldots, n$.

We denote by $v_k = v_k(t,\xi,q)$, $g_k = g_k(t,\xi,q)$ and $v_{0,k} = v_{0,k}(\xi,q)$, $\xi \in S\big(\mathbb{R},\mathbb{R}^d\big)$, $q \in \mathbb{R}^n$, the Laplace transforms of the generalized functions $u_k = u_k(t,\omega,x)$, $f_k = f_k(t,\omega,x)$ and the initial condition $u_{0,k} = u_{0,k}(\omega,x)$, respectively, for $k = 1,\dots,n$. Applying the Laplace transform to the system (16) we obtain

$$\begin{cases} \partial_t v_k - \sum_{j=1}^{n} q_j v_j v_k = \nu q^2 v_k + g_k v_k \\ v_k(0,q) = v_{0,k}(q). \end{cases} \tag{17}$$

Changing the variables, $S_k = \frac{1}{v_k}$, $k = 1,\dots,n$, the system (17) is equivalent to the following:

$$\begin{cases} -\partial_t S_k - \sum_{j=1}^{n} q_j \dfrac{S_k}{S_j} = (\nu q^2 + g_k) S_k \\ S_k(0,\xi,q) = S_{0,k}(\xi,q). \end{cases} \tag{18}$$

We denote $S_1 \cdots \check{S}_k \cdots S_n = S_1 \cdots S_{k-1} S_{k+1} \cdots S_n$, for $k = 1,\dots,n$. If we multiply the first equation of the system (18) by $S_1 \cdots \check{S}_k \cdots S_n$, we deduce

$$(-\partial_t S_k) S_1 \cdots \check{S}_k \cdots S_n - \sum_{j=1}^{n} q_j S_1 \cdots \check{S}_j \cdots S_n = (\nu q^2 + g_k) S_1 \cdots S_n. \tag{19}$$

Let us denote such equation by (E_k), $k = 1,\dots,n$. If we fixe k, then the difference $(E_1) - (E_k)$ becomes

$$\check{S}_1 S_2 \cdots \check{S}_k \cdots S_n (-S_k \partial_t S_1 + S_1 \partial_t S_k) = (g_1 - g_k) S_1 \cdots S_n. \tag{20}$$

After simplification, we divide by $S_1 S_k$ and obtain the equation

$$\frac{\partial_t S_k}{S_k} - \frac{\partial_t S_1}{S_1} = g_1 - g_k \tag{21}$$

which can be integrated. The solution is

$$S_k = \frac{S_{0,k}}{S_{0,1}}\, S_1\, \mathrm{e}^{\int_0^t (g_1(s) - g_k(s)) ds}, \;\; k = 1,\cdots,n \tag{22}$$

where for simplification of notation $g_j(s) = g_j(s,\xi,q)$, $j = 1,\cdots,n$. Then we have the relation

$$\frac{S_k}{S_j} = \frac{S_{0,k}}{S_{0,j}} \mathrm{e}^{\int_0^t (g_j(s)-g_k(s))ds}, \quad k,j = 1,\cdots,n. \tag{23}$$

Introducing the expression of $\frac{S_k}{S_j}$ in (18) we deduce the following linear system of equations

$$\partial_t S_k + \sum_{j=1}^{n} q_j \frac{S_{0,k}}{S_{0,j}} \mathrm{e}^{\int_0^t (g_j(s)-g_k(s))ds} = -(\nu q^2 + g_k) S_k, \quad k = 1,\ldots,n. \tag{24}$$

For any fixed k, the solution of the homogeneous equation is given by

$$S_k(t,q) = \lambda \mathrm{e}^{-\int_0^t (\nu q^2 + g_k(s))ds}, \tag{25}$$

where λ is a constant. Then the solution of (24) is given by the method of variation of constants as

$$\begin{aligned} S_k(t,\xi,q) \;&=\; \lambda \mathrm{e}^{-\int_0^t (\nu q^2 + g_k(s))ds} \\ &\quad - \mathrm{e}^{-\int_0^t (\nu q^2 + g_k(s))ds} \sum_{j=1}^{n} q_j \frac{S_{0,k}}{S_{0,j}} \int_0^t \mathrm{e}^{\int_0^\tau (g_j(s)+\nu q^2)ds} d\tau, \end{aligned} \tag{26}$$

where the constant λ is determined by the initial conditions; $S_k(0,\xi,q) = S_{0,k} = \lambda$. Then (26) may be written as

$$\begin{aligned} S_k(t,\xi,q) \;&=\; S_{0,k}\, \mathrm{e}^{-\int_0^t g_k(s)ds}\, \mathrm{e}^{-\nu q^2 t} \\ &\quad \left(1 - \sum_{j=1}^{n} \frac{q_j}{S_{0,j}} \int_0^t \mathrm{e}^{\int_0^\tau (g_j(s)+\nu q^2)ds} d\tau \right). \end{aligned} \tag{27}$$

Since $S_k(t,\xi,q) = \dfrac{1}{v_k(t,\xi,q)}$, we obtain

$$\begin{aligned} v_k(t,\xi,q) \;&=\; v_{0,k}\, \mathrm{e}^{\int_0^t g_k(s)ds}\, \mathrm{e}^{\nu q^2 t} \\ &\quad \left(1 - \sum_{j=1}^{n} q_j v_{0,j} \int_0^t \mathrm{e}^{\int_0^\tau (g_j(s)+\nu q^2)ds} d\tau \right)^{-1}. \end{aligned} \tag{28}$$

In fact, it is easy to show that for every $t \geq 0$, the function

$$Y(t,q,\xi) = 1 - \sum_{j=1}^{n} q_j v_{0,j}(\xi,q) \int_0^t \mathrm{e}^{\int_0^\tau (g_j(s,\xi,q)+\nu q^2)ds} d\tau$$

belongs to the space $Hol_0(\mathcal{N})$ and satisfy $Y(t,0,0) = 1 \neq 0$. Then there exists $\mathcal{U}$ a neighborhood of $(0,0)$ of $\mathcal{N}$, such that $Y(t,q,\xi) \neq 0$ for every $(\xi,q) \in \mathcal{U}$. Therefore

$$\frac{1}{Y(t,q,\xi)} \in \ Hol_0(\mathcal{N})$$

which implies that

$$v_k(t,q,\xi) \in \ Hol_0(\mathcal{N}).$$

Finally, to obtain the solution of the equation (14) we use the following equalities:

$$\begin{aligned} \mathcal{L}\left\{e^{*\int_0^t f_k(s)ds}\right\} &= e^{\int_0^t g_k(s)ds} \\ \mathcal{L}\left\{\int_0^t e^{*\int_0^\tau f_k(s)ds} * \gamma_{2\nu\tau})d\tau\right\} &= \int_0^t e^{\int_0^\tau g_k(s)ds} e^{\nu q^2 \tau} d\tau \end{aligned}$$

and then $u_k(t,x)$, $k = 1,\ldots,n$ is given by the Laplace inverse transform according to the theorem 2.1, as in (15). ∎

Corollary 3.2 *If the potential $\vec{f}$ in the equation (14) does not depend of the time variable t, i.e., $\vec{f} = \vec{f}(x)$, then the solution is given by*

$$\begin{aligned} u_k(t,x) &= u_{0,k}(x) * e^{*tf_k} * \gamma_{2\nu t} \\ &\quad * \left(\delta_0 + \sum_{j=1}^{n} \partial_j u_{0,j} * \int_0^t e^{*\tau f_j} * \gamma_{2\nu\tau} d\tau\right)^{*-1} \end{aligned}$$

with $k = 1,\ldots,n$. In particular if $f = 0$, the solution has the from

$$u_k(t,x) = u_{0,k}(x) * \gamma_{2\nu t} * \left(\delta_0 + \nabla \cdot u_0 * \int_0^t \gamma_{2\nu\tau} d\tau\right)^{*-1}$$

with $k = 1,\ldots,n$ and $\nabla\cdot$ represents the divergence operator.

4 Stochastic representation of solution of the n-dimensional convolution equation

In the previous section we obtained an explicit solution of the equation (14). Here, we are interested in a stochastic representation of that solution. Let us start with a simple and useful Lemma:

Lemma 4.1 *Let b, $\alpha \in \mathbb{R}$ with $b \neq 0$. We can write*

$$e^{-b+\alpha} = E\Big[\Big(\frac{\alpha}{b}\Big)^{N_b}\Big] \tag{29}$$

where N_b is a random variable with Poisson distribution with intensity b, defined on a probability space (Ω, P), E denoting the mathematical expectation.

Proof. Since N_b is a random variable with Poisson distribution with intensity b, we have

$$\mathrm{E}\big[f(N_b)\big] = \sum_{n=1}^{\infty} f(n)\frac{b^n \mathrm{e}^{-b}}{n!}, \tag{30}$$

which implies

$$\mathrm{E}\Big[\Big(\frac{\alpha}{b}\Big)^{N_b}\Big] = \sum_{n=1}^{\infty}\Big(\frac{\alpha}{b}\Big)^n \frac{b^n \mathrm{e}^{-b}}{n!} = \mathrm{e}^{-b+\alpha}.$$

■

We use the same notations as in Section 3. First we take $f = 0$ in equation (14) and obtain a stochastic representation for the Laplace transfom v of its solution u.

Theorem 4.2 *Let v be the Laplace transform of the solution of equation (14), with $f = 0$, obtained in Theorem 3.1. There are stochastic processes X_t^k, $k = 1, \ldots, n$ such that*

$$v_k(t, \xi, q) = \frac{1}{E\big(X_t^k\big)}.$$

Proof. According to (27), we have

$$S_k(t,\xi,q) = S_{0,k}\mathrm{e}^{-\nu q^2 t} + \sum_{j=1}^{n} \frac{q_j}{\nu q^2}\,\frac{S_{0,k}}{S_{0,j}}\big(\mathrm{e}^{-\nu q^2 t} - 1\big)$$
$$= \mathrm{e}^{-\lambda t}\, S_{0,k}\, \mathrm{e}^{-\nu q^2 t + \lambda t} + \sum_{j=1}^{n} \frac{q_j}{\nu q^2}\,\frac{S_{0,k}}{S_{0,j}}\,\mathrm{e}^{-\lambda' t}\,\mathrm{e}^{-\nu q^2 t + \lambda' t} - \sum_{j=1}^{n} \frac{q_j}{\nu q^2}\,\frac{S_{0,k}}{S_{0,j}}.$$

Using Lemma 4.1, we can write

$$S_k(t,\xi,q) = \mathrm{e}^{-\lambda t}\, S_{0,k}\,\mathrm{E}\left[\left(\frac{\lambda}{\nu q^2}\right)^{N_{\nu q^2 t}}\right] + \sum_{j=1}^{n} \frac{q_j}{\nu q^2}\,\frac{S_{0,k}}{S_{0,j}}\,\mathrm{e}^{-\lambda' t}\,\mathrm{E}\left[\left(\frac{\lambda'}{\nu q^2}\right)^{N_{\nu q^2 t}}\right]$$
$$- \sum_{j=1}^{n} \frac{q_j}{\nu q^2}\,\frac{S_{0,k}}{S_{0,j}}.$$

Therefore

$$S_k(t,\xi,q) = \mathrm{E}\left[\left(\mathrm{e}^{-t}\, S_{0,k} + \sum_{j=1}^{n} \frac{q_j}{\nu q^2}\,\frac{S_{0,k}}{S_{0,j}}\,\mathrm{e}^{-t}\right)\left(\frac{1}{\nu q^2}\right)^{N_{\nu q^2 t}} - \sum_{j=1}^{n} \frac{q_j}{\nu q^2}\,\frac{S_{0,k}}{S_{0,j}}\right].$$

Defining the stochastic processes

$$X_t^k = \left(\mathrm{e}^{-t}\, S_{0,k} + \sum_{j=1}^{n} \frac{q_j}{\nu q^2}\,\frac{S_{0,k}}{S_{0,j}}\,\mathrm{e}^{-t}\right)\left(\frac{1}{\nu q^2}\right)^{N_{\nu q^2 t}} - \sum_{j=1}^{n} \frac{q_j}{\nu q^2}\,\frac{S_{0,k}}{S_{0,j}}$$

for $k = 1, \ldots, n$, the result of the theorem follows. ■

Remark 4.3 *We would like to stress that the representation in Theorem 3.1 for through the process* x_{t^k}*,* $k = 1, \ldots, n$ *could also be obtained by another process, namely Brownian motion, in an easy way. Nevertheless, since in the next theorem we would like to extend the result for* $f \neq 0$ *we then keep that aproach.*

Let us denote by η^{λ} an exponential random variable with parameter λ defined on a probability space.

Theorem 4.4 *Let v be the Laplace transform of solution of equation (14) obtained in Theorem 3.1. There are stochastic processes X_t^k, Y_t^k, $k = 1, \ldots, n$ such that*

$$v_k(t, \xi, q) = \left[E\left(S_{0,k} X_t^k - \frac{1}{\nu q^2} \sum_{j=1}^{n} \frac{q_j}{S_{0,j}} X_t^k Y_t^k \right) \right]^{-1}. \tag{31}$$

The processes X_t^k, Y_t^k are given explicitly by (33) and (34) below.

Proof. Since $v_k = \frac{1}{S_k}$, with

$$\begin{aligned} S_k(t, \xi, q) \;&=\; S_{0,k} \mathrm{e}^{-\int_0^t (\nu q^2 + g_k(s)) ds} \\ &\quad - \sum_{j=1}^{n} q_j \frac{S_{0,k}}{S_{0,j}} \mathrm{e}^{-\int_0^t (\nu q^2 + g_k(s)) ds} \int_0^t \mathrm{e}^{\int_0^\tau (g_j(s) + \nu q^2) ds} d\tau, \end{aligned} \tag{32}$$

and $S_{0,k} = \frac{1}{v_{0,k}}$. Denoting $G_k(s) = \int_0^s g_k(\tau)\, d\tau$, we may write (32) as

$$\begin{aligned} S_k(t, \xi, q) \;&=\; S_{0,k} \mathrm{e}^{-G_k(t) - \nu q^2 t} \\ &\quad - \sum_{j=1}^{n} q_j \frac{S_{0,k}}{S_{0,j}} \mathrm{e}^{-G_k(t) - \nu q^2 t} \int_0^t \mathrm{e}^{G_j(\tau) + \nu q^2 \tau}\, d\tau. \end{aligned}$$

Using Lemma 4.1

$$\mathrm{e}^{-G_k(t) - \nu q^2 t} = \mathrm{E}\left[\left(\frac{-G_k(t)}{\nu q^2 t} \right)^{N_{\nu q^2 t}} \right].$$

We define the stochastic processes

$$X_t^k := \left(\frac{-G_k(t)}{\nu q^2 t} \right)^{N_{\nu q^2 t}}. \tag{33}$$

Now, we consider a random variable $\eta^{\nu q^2}$ with exponential distribution with parameter νq^2 which is independent of $\mathcal{N}_{\nu q^2 t}$, therefore

$$\mathrm{E}\left[\chi_{\{\eta^{\nu q^2} < t\}} f(\eta^{\nu q^2}) \right] = \int_0^t \nu q^2 f(\tau) \mathrm{e}^{-\nu q^2 \tau}\, d\tau.$$

We obtain

$$\sum_{j=1}^{n} q_j \frac{S_{0,k}}{S_{0,j}} \mathrm{e}^{-G_k(t)-\nu q^2 t} \int_0^t \mathrm{e}^{G_j(\tau)+\nu q^2 \tau}\, d\tau$$

$$= \sum_{j=1}^{n} \frac{q_j\, S_{0,k}}{\nu q^2\, S_{0,j}} \mathrm{e}^{-G_k(t)-\nu q^2 t} \int_0^t \mathrm{e}^{G_j(\tau)+2\nu q^2 \tau}\, \nu q^2\, \mathrm{e}^{-\nu q^2 \tau}\, d\tau$$

$$= \sum_{j=1}^{n} \frac{q_j\, S_{0,k}}{\nu q^2\, S_{0,j}} \mathrm{E}\big(X_t^k\big)\mathrm{E}\big(Y_t^j\big)$$

where

$$Y_t^j := \exp\big(2\nu q^2 \eta^{\nu q^2} + G_j(\eta^{\nu q^2})\big)\chi_{\{\eta^{\nu q^2} < t\}}. \tag{34}$$

Assuming that $\eta^{\nu q^2}$ is independent of the Poisson random variable $N_{\nu q^2 t}$, we deduce

$$S_k(t,\xi,q) = \mathrm{E}\bigg(S_{0,k}\, X_t^k - \sum_{j=1}^{n} \frac{q_j\, S_{0,k}}{\nu q^2\, S_{0,j}} X_t^k\, Y_t^j\bigg), \tag{35}$$

and the result of the theorem follows. ■

Acknowledgment

We thank Martin Grothaus for useful discussions. Financial support by GRICES, Portugal/Tunisia, 2004 and FCT, POCTI - Programa Operacional Ciência, Tecnologia e Inovação, FEDER/POCTI-SFA-1-219 are gratefully acknowledged.

References

[1] M. Ben Chrouda, M. El Oued, and H. Ouerdiane; Convolution calculus and applications to stochastic differential equations. *Soochow J. Math.*, 28(4): 375–388, 2002.

[2] Yu. M. Berezansky and Yu. G. Kondratiev; *Spectral Methods in Infinite-Dimensional Analysis*, volume 1. Kluwer Academic Publishers, Dordrecht, 1995.

[3] J. M. Burgers; *The Nonlinear Diffusion Equation.* D. Reidel Publishing Company, Dordrecht, Holland, 1974.

[4] J. D. Callen, C. C. Hegana, E. D. Held, T. A. Gianakon, S. E. Kruger, and C. R. Sovinec; Nonlocal closures for plasma fluid simulations. *Phys. Plasmas*, 11(5): 2419–2426, 2004.

[5] Weinan E, K. Khanin, A. Mazel, and Ya. Sinai; Invariant measures for Burgers equation with stochastic forcing. *Ann. of Math. (2)*, 151(3): 877–960, 2000.

[6] R. Gannoun, R. Hachaichi, P. Kree, and H. Ouerdiane; Division de fonctions holomorphes a croissance θ-exponentielle. Technical Report E 00-01-04, BiBoS University of Bielefeld, 2000.

[7] R. Gannoun, R. Hachaichi, H. Ouerdiane, and A. Rezgui; Un théoréme de dualité entre espaces de fonctions holomorphes à croissance exponentielle. *J. Funct. Anal.*, 171(1): 1–14, 2000.

[8] T. Hida, H. H. Kuo, J. Potthoff, and L. Streit; *White Noise. An Infinite Dimensional Calculus.* Kluwer Academic Publishers, Dordrecht, 1993.

[9] H. Ouerdiane; Infinite dimensional entire functions and applications to stochastic differential equations. *Not. S. Afr. Math. Soc.*, 35(1): 23–45, 2004.

[10] H. Ouerdiane and J. Silva; On the heat equation with positive generalized stochastic process potential. *Methods Funct. Anal. Topology*, 10(3): 54–63, 2004.

[11] H. Ouerdiane and J. L. Silva; Generaliezed Feymann-Kac formula with stochastic potential. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 5(2): 1–13, 2002.

[12] A. J. Roberts; Planform evolution in convection—an embedded centre manifold. *J. Austral. Math. Soc. Ser. B*, 34(2): 174–198, 1992.

[13] A. J. Roberts; The swift-Hohenberg equation requires non-local modifications to model spatial pattern evolution of physical problems. Technical report, arXiv:patt-sol/9412002, 2002.

[14] P. Ván; Weakly nonlocal continuum theories of granular media: restrictions from the second law. *International Journal of Solids and Structures*, 41(21): 5921–5927, 2004.

[15] P. Ván and T. Fülöp; Weakly nonlocal fluid mechanics - the Schrödinger equation. Technical report, arXiv:quant-ph/0304062, 2004.

[16] E. C. Waymire; Probability and incompressible Navier-Stokes equations: An overview of some recent developments, *Probability Surveys* 2 : 1–32, 2005.