On the heat equation with positive generalized stochastic process potential

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Abstract

In this paper we study the stochastic heat equation with potential and initial condition as generalized stochastic processes. For positive generalized stochastic process potential $(V(t))_{t\geq 0}$ and initial condition f the solution is given as a convergent series of integrals. Our approach is based on the convolution calculus on a suitable distribution space.

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1 Introduction

The aim of this work is to study the solution of the following Cauchy problem corresponding to the heat equation

$$\begin{cases} \frac{\partial}{\partial t}X(t,x,\omega) = a\Delta X(t,x,\omega) + X(t,x,\omega) * V(t,x,\omega) \\ X(0,x,\omega) = f(x,\omega), \end{cases} \tag{1}$$

where $a \in \mathbb{R}_+$, $t \in [0,\infty)$ is the time parameter, $x = (x_1,\dots,x_r) \in \mathbb{R}^r$ is the spatial variable, $r \in \mathbb{N}$, $\Delta = \sum_{i=1}^{r} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian on $\mathbb{R}^r$. The stochastic vector variable $\omega = (\omega_1,\dots,\omega_d)$ is an the tempered Schwartz distribution space $S'_d := S'(\mathbb{R},\mathbb{R}^d)$ with the standard Gaussian measure, $d \in \mathbb{N}$, $*$ is the convolution product between generalized functions (see Subsection 2.2 below) and the initial condition f as well as the potential V are generalized stochastic processes.

The Cauchy problem (1) was analyzed by many authors from different point of view, see e.g., [1], [7], and references therein. Often in the literature is used the Wick product $\diamondsuit$ (see [8] for this notion) instead of convolution product $*$ proposed here.

The motivation to study these equations in such general framework is due to the fact that usually we have insufficient information on the parameters values of the system. In some cases these parameters may be very complicated because they are influenced by the surrounding, the medium or fluctuate due to external or internal random force.

Recently Ouerdiane et al. [14] obtained the solution of (1) in terms of the convolution exponential as a well defined generalized function in a suitable distribution space, see Theorem 3.1 below. The main result of this paper is to prove that for positive generalized stochastic process $V = (V(t))_{t\geq 0}$ and initial condition f the solution is given in terms of a convergent series of integrals. For time independent deterministic potential V and initial condition f the solution is in agreement with the known from the literature, e.g., [1], cf. Proposition 3.5 below.

The starting point is the following Gelfand triple

$$\mathcal{F}'_\theta(\mathcal{N}') \supset L^2(\mathcal{M}',\gamma) \supset \mathcal{F}_\theta(\mathcal{N}'), \tag{2}$$

where $\mathcal{N}'$ is a the dual of a complex nuclear Fréchet space $\mathcal{N}$, θ is a Young function (see definition in Section 2), γ is the usual Gaussian measure on $\mathcal{M}'$ which corresponds to the real part of $\mathcal{N}'$. The test function space $\mathcal{F}_\theta(\mathcal{N}')$ is defined as the space of all holomorphic functions on $\mathcal{N}'$ with an exponential growth condition of order θ. The generalized function space $\mathcal{F}'_\theta(\mathcal{N}')$ represents the topological dual of $\mathcal{F}_\theta(\mathcal{N}')$. In the following we will choose the complex nuclear space $\mathcal{N} = (S_d \times \mathbb{R}^r)_{\mathbb{C}}$, the complexification of the real nuclear space $S_d \times \mathbb{R}^r$, which is adapted to our situation.

Using the Laplace transform $\mathcal{L}$ (which is a topological isomorphism, cf. Theorem 2.1 below) we may define the convolution of two generalized functions $\Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}')$ by

$$\Phi * \Psi = \mathcal{L}^{-1}(\mathcal{L}\Phi \cdot \mathcal{L}\Psi).$$

The convolution exponential of Φ denoted by $\exp^* \Phi$ is then introduced as an element in $\mathcal{F}'_\varphi(\mathcal{N}')$, where the Young function $\varphi = (e^{\theta^*})^*$ and

$$\theta^*(x) := \sup_{y \geq 0}(yx - \theta(y)) \tag{3}$$

denotes the polar function associated to θ, see e.g., [9].

For positive generalized stochastic process $V = (V(t))_{t\geq 0}$ there exists a family of Radon measures $\mu = (\mu_t)_{t\geq 0}$ (see e.g., [13]) on $\mathcal{M}'$ which represents V and, therefore, the Fourier transform of μ_t, $t \geq 0$ is given by

$$\hat{\mu}_t(\xi) = \langle\langle V(t), \exp(i\xi)\rangle\rangle = \int_{\mathcal{M}'} \exp(i\langle y, \xi\rangle) d\mu_t(y),$$

where $\langle\langle \cdot, \cdot \rangle\rangle$ denotes the duality between $\mathcal{F}'_\theta(\mathcal{N}')$ and $\mathcal{F}_\theta(\mathcal{N}')$ which corresponds to the extension of the inner product of $L^2(\mathcal{M}', \gamma)$. Under these hypothesis we prove that for any test function $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ and all $u \in \mathcal{N}'$ we have

$$\begin{aligned} & \left(\exp^* \left(\int_0^t V_s ds\right) * \varphi\right)(u) \\ = \quad & \varphi(u) + \sum_{n=1}^\infty \frac{1}{n!} \int_{[0,t]^n} \int_{(\mathcal{M}')^n} \varphi(u + y_1 + \ldots + y_n) \prod_{i=1}^n d\mu_{s_i}(y_i) ds_i \end{aligned}$$

which connects the convolution calculus and convergent series of integrals. We will use this equality to write the solution $X(t, x)$ of (1) for deterministic potential V and suitable choice of φ as

$$\begin{aligned} X(t,x) &= (4a\pi t)^{-r/2} \int_{\mathbb{R}^r} f(y) e^{\frac{|x-y|^2}{4at}} dy \\ &+ (4a\pi t)^{-r/2} \sum_{n=1}^\infty \frac{1}{n!} \int_{[0,t]^n} \int_{(\mathbb{R}^r)^n} \int_{\mathbb{R}^r} f(y) e^{\frac{|x+y_1+\ldots+y_n-y|^2}{4at}} dy \prod_{i=1}^n d\mu_{s_i}(y_i) ds_i. \end{aligned}$$

We would like to mention that the positivity of the potential V implies the following property for μ: for each $t \geq 0$ there exists $n \in \mathbb{N}$, $m > 0$ with $\mu_t(\mathcal{M}_{-n}) = 1$ and μ_t satisfies the integrability condition,

$$\int_{\mathcal{M}_{-n}} \exp(\theta(m|y|_{-n})) d\mu_t(y) < \infty. \tag{4}$$

If V is deterministic and time independent, then the corresponding measure μ which verify (4) implies that V belongs to the so-called Albeverio-Høegh-Krohn class, see [1]. This class of potentials was studied by Asai et al. [2] and Kuna et al. [10] for the Schrödinger equation in connection with Feynman integrals. Our method may also be applied to solve the Cauchy problem corresponding to the Schrödinger equation if we replace a by $i\frac{\hbar}{2m}$, where $\hbar$ is the Plank's constant divided by 2π and m is the mass of the non relativistic particle, see Remark 3.8 for more details.

Finally, we would like to mention that our method also applies to smooth initial conditions f and potential V. In fact, any test function $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ or any $h \in L^2(\mathcal{M}', \gamma)$ may be considered for initial condition f or to play the role of the potential V because, in both cases we have an element in $\mathcal{F}'_\theta(\mathcal{N}')$ due to the triple (2). In that case the convolution product turns into the usual convolution product with respect to γ and the dual pairing is simply the inner product in $L^2(\mathcal{M}', \gamma)$, see Remark 3.9-2.

2 Preliminaries

2.1 Test and generalized functions spaces

In this section we introduce the framework need later on. The starting point is the real Hilbert space $\mathcal{H} = L^2(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r$, $d, r \in \mathbb{N}$ with scalar product $(\cdot, \cdot)$ and norm $|\cdot|$. More precisely, if $(f, x) = ((f_1, \dots, f_d), (x_1, \dots, x_r)) \in \mathcal{H}$, then

$$|(f, x)|^2 := \sum_{i=1}^{d} \int_{\mathbb{R}} f_i^2(u)du + \sum_{i=1}^{r} x_i^2 = |f|^2_{L^2(\mathbb{R},\mathbb{R}^d)} + |x|^2_{\mathbb{R}^r}.$$

Let us consider the real nuclear triplet

$$\mathcal{M}' = S'(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r \supset \mathcal{H} \supset S(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r = \mathcal{M}. \tag{5}$$

The pairing $\langle \cdot, \cdot \rangle$ between $\mathcal{M}'$ and $\mathcal{M}$ is given in terms of the scalar product in $\mathcal{H}$, i.e., $\langle (\omega, x), (\xi, p) \rangle := (\omega, \xi)_{L^2(\mathbb{R},\mathbb{R}^d)} + (x, p)_{\mathbb{R}^r}$, $(\omega, x) \in \mathcal{M}'$ and $(\xi, p) \in \mathcal{M}$. Since $\mathcal{M}$ is a Fréchet nuclear space, then it can be represented as

$$\mathcal{M} = \bigcap_{n=0}^{\infty} S_n(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r = \bigcap_{n=0}^{\infty} \mathcal{M}_n,$$

where $S_n(\mathbb{R}, \mathbb{R}^d) \times \mathbb{R}^r$ is a Hilbert space with norm square given by $|\cdot|_n^2 + |\cdot|^2_{\mathbb{R}^r}$, see e.g., [6] or [4] and references therein. We will consider the complexification of the triple (5) and denote it by

$$\mathcal{N}' \supset \mathcal{Z} \supset \mathcal{N}, \tag{6}$$

where $\mathcal{N} = \mathcal{M} + i\mathcal{M}$ and $\mathcal{Z} = \mathcal{H} + i\mathcal{H}$. On $\mathcal{M}'$ we have the standard Gaussian measure γ given by Minlos's theorem via its characteristic functional: for every $(\xi, p) \in \mathcal{M}$

$$C_\gamma(\xi, p) = \int_{\mathcal{M}'} \exp(i\langle(\omega, x), (\xi, p)\rangle) d\gamma((\omega, x)) = \exp(-\frac{1}{2}(|\xi|^2 + |p|^2)).$$

In order to solve the Cauchy problem (1) we need to introduce an appropriate space of generalized functions. We borrow this construction from [11]. Let $\theta = (\theta_1, \theta_2) : \mathbb{R}_+^2 \to \mathbb{R}$, $(t_1, t_2) \mapsto \theta_1(t_1) + \theta_2(t_2)$ where θ_1, θ_2 are two Young functions, i.e., $\theta_i : \mathbb{R}_+ \to \mathbb{R}_+$ continuous convex strictly increasing function and

$$\lim_{t\to\infty} \frac{\theta_i(t)}{t} = \infty, \quad \theta_i(0) = 0, \quad i = 1, 2.$$

For every pair $m = (m_1, m_2)$ with $m_1, m_2 \in]0, \infty[$, we define the Banach space $\mathcal{F}_{\theta,m}(\mathcal{N}_{-n})$, $n \in \mathbb{N}$ by

$$\mathcal{F}_{\theta,m}(\mathcal{N}_{-n}) := \{f : \mathcal{N}_{-n} \to \mathbb{C}, \text{ entire}, \ \|f\|_{\theta,m,n} = \sup_{z\in\mathcal{N}_{-n}} |f(z)| \exp(-\theta(m|z|_{-n})) < \infty\},$$

where for each $z = (\omega, x)$ we have $\theta(m|z|_{-n}) := \theta_1(m_1|\omega|_{-n}) + \theta_2(m_2|x|)$. Now we consider as test function space the space of entire functions on $\mathcal{N}'$ of (θ_1, θ_2)-exponential growth and minimal type

$$\mathcal{F}_\theta(\mathcal{N}') = \bigcap_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0} \mathcal{F}_{\theta,m}(\mathcal{N}_{-n}),$$

endowed with the projective limit topology. We would like to construct the triple of the complex Hilbert space $L^2(\mathcal{M}', \gamma)$ by $\mathcal{F}_\theta(\mathcal{N}')$. To this end we need another condition on the pair of Young functions (θ_1, θ_2). Namely,

$$\lim_{t\to\infty} \frac{\theta_i(t)}{t^2} < \infty, \quad i = 1, 2. \tag{7}$$

This is enough to obtain the following Gelfand triple

$$\mathcal{F}_\theta'(\mathcal{N}') \supset L^2(\mathcal{M}', \gamma) \supset \mathcal{F}_\theta(\mathcal{N}'), \tag{8}$$

where $\mathcal{F}'_\theta(\mathcal{N}')$ is the topological dual of $\mathcal{F}_\theta(\mathcal{N}')$ with respect to $L^2(\mathcal{M}', \gamma)$ endowed with the inductive limit topology.

In applications it is very important to have the characterization of generalized functions from $\mathcal{F}'_\theta(\mathcal{N}')$. First we define the Laplace transform of an element in $\mathcal{F}'_\theta(\mathcal{N}')$. For every fixed element $(\xi, p) \in \mathcal{N}$ the exponential function $\exp((\xi, p))$ is a well defined element in $\mathcal{F}_\theta(\mathcal{N}')$, see [5]. The Laplace transform $\mathcal{L}$ of a generalized function $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ is defined by

$$\hat{\Phi}(\xi, p) := (\mathcal{L}\Phi)(\xi, p) := \langle\langle \Phi, \exp((\xi, p)) \rangle\rangle. \tag{9}$$

We are ready to state to characterization theorem (see e.g., [5] for the proof) which is the main tool in our further consideration.

Theorem 2.1 *1. The Laplace transform is a topological isomorphism between $\mathcal{F}'_\theta(\mathcal{N}')$ and the space $\mathcal{G}_{\theta^*}(\mathcal{N})$, where $\mathcal{G}_{\theta^*}(\mathcal{N})$ is defined by*

$$\mathcal{G}_{\theta^*}(\mathcal{N}) = \bigcup_{m \in (\mathbb{R}_+^*)^2, n \in \mathbb{N}_0} \mathcal{G}_{\theta^*, m}(\mathcal{N}_n),$$

and $\mathcal{G}_{\theta^*,m}(\mathcal{N}_n)$ is the space of entire functions on $\mathcal{N}_n$ with the following θ-exponential growth condition

$$\mathcal{G}_{\theta^*,m}(\mathcal{N}_n) \ni g, \ |g(\xi, p)| \leq k \exp(\theta_1^*(m_1 |\xi|_n) + \theta_2^*(m_2 |p|)), \ (\xi, p) \in \mathcal{N}_n.$$

2. The Laplace transform is a topological isomorphism between $\mathcal{F}_\theta(\mathcal{N}')$ and itself.

2.2 The Convolution Product $*$

It is well known that in infinite dimensional complex analysis the convolution operator on a general function space $\mathcal{F}$ is defined as a continuous operator which commutes with the translation operator. Let us define the convolution between a generalized and a test function. Let $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ be given, then the convolution $\Phi * \varphi$ is defined by

$$(\Phi * \varphi)(\omega, x) := \langle\langle \Phi, \tau_{-(\omega,x)}\varphi \rangle\rangle,$$

where $\tau_{-(\omega,x)}$ is the translation operator, i.e.,

$$(\tau_{-(\omega,x)}\varphi)(\eta, y) := \varphi(\omega + \eta, x + y).$$

It is not hard the see that $\Phi * \varphi \in \mathcal{F}_\theta(\mathcal{N}')$. The convolution product is given in terms of the dual pairing as $(\Phi * \varphi)(0,0) = \langle\langle \Phi, \varphi \rangle\rangle$ for any $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$.

We can generalize the above convolution product for generalized functions as follows. Let $\Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}')$ be given, then $\Phi * \Psi$ is defined by

$$\langle\langle \Phi * \Psi, \varphi \rangle\rangle := \langle\langle \Phi, \Psi * \varphi \rangle\rangle,\ \forall \varphi \in \mathcal{F}_\theta(\mathcal{N}'). \tag{10}$$

This definition of convolution product for generalized functions will be used on Section 3 in order to write the solution of the stochastic heat equation in (1). We have the following equality, see [14], Proposition 3.3:

$$\Phi * \exp((\xi, p)) = (\mathcal{L}\Phi)(\xi, p) \exp((\xi, p)),\ (\xi, p) \in \mathcal{N}.$$

As a consequence of the above equality and definition (10) we obtain

$$\mathcal{L}(\Phi * \Psi) = \mathcal{L}\Phi\mathcal{L}\Psi,\ \Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}') \tag{11}$$

which says that the Laplace transform maps the convolution product of $\mathcal{F}'_\theta(\mathcal{N}')$ into the usual pointwise product in the algebra of functions $\mathcal{G}_{\theta^*}(\mathcal{N})$. Therefore we may use Theorem 2.1 to define convolution product between two generalized functions as

$$\Phi * \Psi = \mathcal{L}^{-1}(\mathcal{L}\Phi\mathcal{L}\Psi).$$

This allows us to introduce the convolution exponential of a generalized function. In fact, for every $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ we may easily check that $\exp(\mathcal{L}\Phi) \in \mathcal{G}_{e^{\theta^*}}(\mathcal{N})$. Using the inverse Laplace transform and the fact that any Young function θ verify the property $(\theta^*)^* = \theta$ we obtain that $\mathcal{L}^{-1}(\mathcal{G}_{e^{\theta^*}}(\mathcal{N})) = \mathcal{F}'_{(e^{\theta^*})^*}(\mathcal{N}')$. Now we give the definition of the convolution exponential of $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$, denoted by $\exp^* \Phi$

$$\exp^* \Phi := \mathcal{L}^{-1}(\exp(\mathcal{L}\Phi)).$$

Notice that $\exp^* \Phi$ is well a defined element in $\mathcal{F}'_{(e^{\theta^*})^*}(\mathcal{N}')$ and therefore the distribution $\exp^* \Phi$ is given in terms of a convergent series

$$\exp^* \Phi = \delta_0 + \sum_{n=1}^{\infty} \frac{1}{n!} \Phi^{*n}, \tag{12}$$

where Φ^{*n} is the convolution of Φ with itself n times, $\Phi^{*0} := \delta_0$ by convention with δ_0 denoting the Dirac distribution at 0. We refer to [3] for more details concerning convolution product on $\mathcal{F}'_\theta(\mathcal{N}')$.

3 Applications to the heat equation

A one parameter generalized stochastic process with values in $\mathcal{F}'_\theta(\mathcal{N}')$ is a family of distributions $\{\Phi(t),\ t \geq 0\} \subset \mathcal{F}'_\theta(\mathcal{N}')$. The process $\Phi(t)$ is said to be continuous if the map $t \mapsto \Phi(t)$ is continuous. For a given continuous generalized stochastic process $(X(t))_{t\geq 0}$ we define the generalized stochastic process

$$Y(t,x,\omega) = \int_0^t X(s,x,\omega)ds \in \mathcal{F}'_\theta(\mathcal{N}')$$

by

$$\mathcal{L}\left(\int_0^t X(s,x,\omega)ds\right)(\xi,p) := \int_0^t \mathcal{L}X(s,p,\xi)ds. \tag{13}$$

The process $Y(t,x,\omega)$ is differentiable and we have $\frac{\partial}{\partial t}Y(t,x,\omega) = X(t,x,\omega)$. The details of the proof can be seen in [14], Proposition 4.11. The main result in [14] is stated in the following theorem.

Theorem 3.1 *1. The Cauchy problem (1) has an unique solution $X(t)$ which is a generalized $\mathcal{F}'_\beta(\mathcal{N}')$-valued stochastic process, where the Young function β is given by $\beta = (e^{\theta^*})^*$. Moreover, the solution $X(t)$ is given explicitly by*

$$X(t,\omega,x) = f(\omega,x) * \exp^*\left(\int_0^t V(s)(\omega,x)ds\right) * \gamma_{2at}, \tag{14}$$

where γ_{2at} is Gaussian measure on $\mathbb{R}^r$ with variance $2at$.

2. If the potential V and the initial condition f do not depend on the random parameter ω then the solution of (1) is given by

$$X(t,x) = (g(t,\cdot) * \gamma_{2at})(x), \tag{15}$$

where g is equal to

$$g(t,x) = f(x)\exp\left(\int_0^t V(s,x)ds\right).$$

We are now going to write the solution of the Cauchy problem (1) as a limit of convergent series of integrals. To this end, we choose the potential $V = (V(t))_{t\geq 0}$

as a positive generalized stochastic process represented by the family of Radon measures $(\mu_t)_{t\geq 0}$, i.e., for any $t \geq 0$

$$\langle\langle V(t), \varphi \rangle\rangle = \int_{\mathcal{M}'} \varphi(y) d\mu_t(y), \quad \varphi \in \mathcal{F}_\theta(\mathcal{N}').$$

Moreover the measure μ_t verify the following integrability condition: there exists $n \in \mathbb{N}$ and $m > 0$ with $\mu_t(\mathcal{M}_{-n}) = 1$ such that

$$\int_{\mathcal{M}_{-n}} \exp(\theta(m|y|_{-n})) d\mu_t(y) < \infty. \tag{16}$$

Lemma 3.2 *For each Radon measure μ on $\mathcal{M}'$ verifying (16) and all $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ we have for any $u = (x, \omega) \in \mathcal{N}' = \mathcal{M}' + i\mathcal{M}'$*

$$((\exp^* \mu) * \varphi)(u) = \varphi(u) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathcal{M}')^n} \varphi(u + y_1 + \ldots + y_n) d\mu(y_1) \ldots d\mu(y_n). \tag{17}$$

Proof. First we compute $\mu^{*n} * \varphi$, for any $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ and $n = 2$. Hence if $u \in \mathcal{N}'$ we have

$$\begin{aligned}
((\mu * \mu) * \varphi)(u) &= (\mu * (\mu * \varphi))(u) \\
&= \langle\langle \mu, \tau_{-u}(\mu * \varphi) \rangle\rangle \\
&= \int_{\mathcal{M}'} \tau_{-u}(\mu * \varphi)(y_1) d\mu(y_1) \\
&= \int_{\mathcal{M}'} (\mu * \varphi)(u + y_1) d\mu(y_1) \\
&= \int_{\mathcal{M}'} \left(\int_{\mathcal{M}'} \varphi(u + y_1 + y_2) d\mu(y_2) \right) d\mu(y_1).
\end{aligned}$$

Now, using iteratively this procedure on the equality (12) we obtain for every $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ and any $u \in \mathcal{N}'$

$$\begin{aligned}
&((\exp^* \mu) * \varphi)(u) \qquad (18)\\
&= \varphi(u) + \sum_{n=1}^{\infty} \frac{1}{n!} (\mu^{*n} * \varphi)(u) \\
&= \varphi(u) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathcal{M}')^*} \varphi(u + y_1 + \ldots + y_n) d\mu(y_1) \ldots d\mu(y_n).
\end{aligned}$$

This proves the desired result. ■

Lemma 3.3 *Let $(V(s))_{s\geq 0} \subset \mathcal{F}'_\theta(\mathcal{N}')$ be a positive generalized stochastic process represented by the family of measures $(\mu_s)_{s\geq 0}$. Then for any $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ we have*

$$\left\langle\!\!\left\langle \int_0^t V(s)ds, \varphi \right\rangle\!\!\right\rangle = \int_0^t \langle\!\langle V(s), \varphi\rangle\!\rangle ds \qquad (19)$$
$$= \int_0^t \left(\int_{\mathcal{M}'} \varphi(y) d\mu_s(y) \right) ds,$$

Moreover, we have

$$\left\langle\!\!\left\langle \exp^* \left(\int_0^t V(s)ds \right), \varphi \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \exp^* \left(\int_0^t \mu_s ds \right), \varphi \right\rangle\!\!\right\rangle. \qquad (20)$$

Proof. In fact equality (19) is nothing but the definition (13) with $\varphi = \exp((\xi, p))$. Therefore by a limit procedure we get the required result (19) for general test function $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ since the set of $\exp((\xi, p))$, $(\xi, p) \in \mathcal{N}$ is total in $\mathcal{F}_\theta(\mathcal{N}')$.

To prove equality (20) we proceed in two steps: first we notice that for every $s \geq 0$ $V(s) * V(s)$ is represented by $\mu_s * \mu_s$ which follows from the following calculation with $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$

$$\begin{aligned} \langle\!\langle V(s) * V(s), \varphi\rangle\!\rangle &= \langle\!\langle V(s), V(s) * \varphi\rangle\!\rangle \\ &= \int_{\mathcal{M}'} (V(s) * \varphi)(x) d\mu_s(x) \\ &= \int_{\mathcal{M}'} \left(\int_{\mathcal{M}'} \varphi(x+y) d\mu_s(y) \right) d\mu_s(x) \\ &= \langle\!\langle \mu_s * \mu_s, \varphi\rangle\!\rangle. \end{aligned}$$

Iterating this process we obtain

$$\langle\!\langle \exp^* V(s), \varphi\rangle\!\rangle = \langle\!\langle \exp^* \mu_s, \varphi\rangle\!\rangle. \qquad (21)$$

Then equality (20) is a consequence of (19) and (21). ■

We now use these two lemmas to derive the following corollary.

Corollary 3.4 *Let $(V(s))_{s\geq 0} \subset \mathcal{F}'_\theta(\mathcal{N}')$ be a positive generalized stochastic process represented by the family of measures $(\mu_s)_{s\geq 0}$. Then for any test function*

$\varphi \in \mathcal{F}_\theta(\mathcal{N}')$, $u \in \mathcal{N}'$ *holds*

$$\begin{aligned} &\left(\exp^*\left(\int_0^t V(s)ds\right) * \varphi\right)(u) \qquad (22)\\ = \quad & \varphi(u) + \sum_{n=1}^\infty \frac{1}{n!}\int_{[0,t]^n}\int_{(\mathcal{M}')^n} \varphi(u+y_1+\ldots+y_n)\prod_{i=1}^n d\mu_{s_i}(y_i)ds_i. \end{aligned}$$

We are now ready to write the solution (14) of the Cauchy problem (1) as a convergent series of integrals. We will apply the preceding corollary with φ of the following form

$$\varphi_t(x) = (\gamma_{2at} * f)(x) = (4a\pi t)^{-r/2}\int_{\mathbb{R}^r} f(y)e^{-\frac{|x-y|^2}{4at}}dy, \quad x \in \mathbb{R}^r,$$

where the initial condition f is a given function.

Proposition 3.5 *Let V, f be deterministic functions. The solution of the Cauchy problem (1) admits the following representation*

$$\begin{aligned} & X_t(x) \\ = \quad & \varphi_t(x) + \sum_{n=1}^\infty \frac{1}{n!}\int_{[0,t]^n}\int_{(\mathbb{R}^r)^n} \varphi_t(x+y_1+\ldots+y_n)\prod_{i=1}^n d\mu_{s_i}(y_i)ds_i \\ = \quad & (4a\pi t)^{-r/2}\int_{\mathbb{R}^r} f(y)e^{-\frac{|x-y|^2}{4at}}dy \\ & +(4a\pi t)^{-r/2}\sum_{n=1}^\infty \frac{1}{n!}\int_{[0,t]^n}\int_{(\mathbb{R}^r)^n}\int_{\mathbb{R}^r} f(y)e^{-\frac{|x+y_1+\ldots+y_n-y|^2}{4at}}dy\prod_{i=1}^n d\mu_{s_i}(y_i)ds_i. \end{aligned}$$

If the potential V is time independent and $r = 1$ then the solution is given by

$$\begin{aligned} X_t(x) \quad = \quad & (4a\pi t)^{-1/2}\int_{\mathbb{R}} f(y)e^{-\frac{|x-y|^2}{4at}}dy \\ & +(4\pi a t)^{-1/2}\sum_{n=1}^\infty \frac{t^n}{n!}\int_{\mathbb{R}^n}\int_{\mathbb{R}} f(y)e^{-\frac{(x+y_1+\ldots+y_n-y)^2}{4at}}dy d\mu(y_1)\ldots d\mu(y_n). \end{aligned}$$

We are now going to obtain the analogous of Proposition 3.5 for initial condition from our generalized function space $\mathcal{F}'_{\theta^*}(\mathcal{N}')$. Before let us define the adjoint of the translation operator applied to a generalized function from $\mathcal{F}'_{\theta^*}(\mathcal{N}')$. For any $x \in \mathcal{N}'$ and $\Phi \in \mathcal{F}'_{\theta^*}(\mathcal{N}')$ we define the generalized function $\tau^*_{-x}\Phi \in \mathcal{F}'_{\theta^*}(\mathcal{N}')$ as

$$\langle\langle \tau^*_{-x}\Phi, \varphi \rangle\rangle = \langle\langle \Phi, \tau_{-x}\varphi \rangle\rangle, \quad \forall \varphi \in \mathcal{F}_{\theta^*}(\mathcal{N}').$$

Now we generalize Lemma 3.2.

Lemma 3.6 *Let μ be a Radon measure on $\mathcal{M}'$ fulfilling condition (16), then for every distribution $\Phi \in \mathcal{F}'_{\theta^*}(\mathcal{N}')$ we have*

$$(\exp^* \mu) * \Phi = \Phi + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Phi d\mu(y_1)\ldots d\mu(y_n), \tag{23}$$

where for every $n = 1, 2, \ldots$ the distribution $\int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Phi d\mu(y_1)\ldots d\mu(y_n)$ is defined for any $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ as

$$\begin{aligned}
&\left\langle\!\!\left\langle \int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Phi d\mu(y_1)\ldots d\mu(y_n), \varphi \right\rangle\!\!\right\rangle \\
= &\int_{(\mathcal{M}')^n} \langle\langle \tau^*_{-y_1-\ldots-y_n}\Phi, \varphi \rangle\rangle d\mu(y_1)\ldots d\mu(y_n) \\
= &\int_{(\mathcal{M}')^n} \langle\langle \Phi, \tau_{-y_1-\ldots-y_n}\varphi \rangle\rangle d\mu(y_1)\ldots d\mu(y_n) \\
= &\int_{(\mathcal{M}')^n} (\Phi * \varphi)(y_1 + \ldots + y_n) d\mu(y_1)\ldots d\mu(y_n).
\end{aligned}$$

Proof. Equality (23) may be derived as follows: for any test function $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ definition (10) gives

$$\langle\langle (\exp^* \mu) * \Phi, \varphi \rangle\rangle = \langle\langle \exp^* \mu, \Phi * \varphi \rangle\rangle.$$

Now we use the relation between the convolution product and dual pairing to obtain

$$\langle\langle \exp^* \mu, \Phi * \varphi \rangle\rangle) = ((\exp^* \mu) * (\Phi * \varphi))(0).$$

Applying Lemma 3.2 with $\Phi * \varphi$ replacing φ yields

$$\begin{aligned} & \langle\langle (\exp^* \mu) * \Phi, \varphi \rangle\rangle \\ = \;& (\Phi * \varphi)(0) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathcal{M}')^n} (\Phi * \varphi)(0 + y_1 + \ldots + y_n) d\mu(y_1) \ldots d\mu(y_n) \\ = \;& \langle\langle \Phi, \varphi \rangle\rangle + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathcal{M}')^n} \langle\langle \Phi, \tau_{-y_1-\ldots-y_n} \varphi \rangle\rangle d\mu(y_1) \ldots d\mu(y_n) \\ = \;& \langle\langle \Phi, \varphi \rangle\rangle + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{(\mathcal{M}')^n} \langle\langle \tau^*_{-y_1-\ldots-y_n} \Phi, \varphi \rangle\rangle d\mu(y_1) \ldots d\mu(y_n). \end{aligned}$$

■

Theorem 3.7 *Let $(V(t))_{t \geq 0}$ be a positive generalized stochastic process represented by the family of Radon measures $(\mu_t)_{t \geq 0}$ on $\mathcal{M}'$ which verify the integrability condition (16). If the initial condition f is a generalized function in $\mathcal{F}'_{\theta^*}(\mathcal{N}')$, then the solution of the Cauchy problem (1) is given by*

$$\left(\exp^* \int_0^t V(s) ds \right) * \Psi = \Psi + \sum_{n=1}^{\infty} \frac{1}{n!} \int_{[0,t]^n} \int_{(\mathcal{M}')^n} \tau^*_{-y_1-\ldots-y_n} \Psi \prod_{i=1}^{n} d\mu_{s_i}(y_i) ds_i,$$

where Ψ is the distribution given by $f * (\gamma_{2at} \otimes \delta_0)$, here γ_{2at} is the Gaussian measure on $\mathbb{R}^r$ with variance $2at$ and δ_0 is the Dirac measure on S'_d.

Proof. The prove is a consequence of (22) and (23). ■

Remark 3.8 *The Cauchy problem corresponding to the Schrödinger equation is*

$$\begin{cases} i\hbar \frac{\partial}{\partial t} X(t,x) = -\frac{\hbar^2}{2m} \Delta X(t,x) + X(t,x) * V(t,x) \\ X(0,x) = f(x). \end{cases}$$

In our framework this corresponds to choose $a = i\frac{\hbar}{2m}$ and interpret the measure γ_{2at} as a generalized function defined for any test function $\varphi \in \mathcal{F}_\theta(\mathbb{C}^r)$ by

$$(\gamma_{2at} * \varphi)(x) = (2\pi i t\hbar/m)^{-r/2} \int_{\mathbb{R}^r} \varphi(y) e^{im\frac{|x-y|^2}{2t\hbar}} dy.$$

Hence the corresponding solution is given by

$$X(t,x) = (2\pi i t\hbar/m)^{-r/2} \int_{\mathbb{R}^r} f(y) \exp\left(\int_0^t V(s,y) ds \right) e^{im\frac{|x-y|^2}{2t\hbar}} dy.$$

Remark 3.9 *1. We would like to mention that the spaces $\mathcal{F}_\theta(\mathcal{N}')$ and its dual $\mathcal{F}'_\theta(\mathcal{N}')$ are independent of the Gaussian measure γ. For another probability measure P on $\mathcal{M}'$ we can construct the analogous Gelfand triple as in (8) changing in an appropriate way the condition on the Young function θ in (7).*

2. If one wants to handle potential not as generalized functions as we do here but as an ordinary function, e.g., $v \in \mathcal{F}_\theta(\mathcal{N}')$, then we may identify v with the generalized function $vP \in \mathcal{F}'_\theta(\mathcal{N}')$. In fact, for any test function $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ we have

$$\langle\langle vP, \varphi\rangle\rangle = \langle\langle P, v\varphi\rangle\rangle = \int_{\mathcal{M}'} v(x)\varphi(x)dP(x)$$

and this obviously defines a linear continuous functional on $\mathcal{F}_\theta(\mathcal{N}')$. Notice that in this case $vP * \varphi$ coincides with the usual convolutions $(v * \varphi)_P$ between functions with respect to the measure P.

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