Marked Gibbs measures via cluster expansion

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Abstract

In this paper we give a sufficiently detailed account on the existence of marked Gibbs measures in the high temperature and low fugacity regime. This is proved for wide class of underlying spaces and potentials such that stability and integrability conditions are satisfied. That is, for state space we take a Riemannian manifold X and a separable Banach space for the mark space. This framework allowed us to cover the loop space. Furthermore we also explain how to extend the construction for more general spaces as e.g., separable standard Borel spaces. The existence theorem for the marked Gibbs measures is based on the method of cluster expansion following the ideas of Ruelle and Malyshev-Minlos, this is developed in great details. We also proved that for finite range potentials the marked Gibbs measure satisfies the DLR equations.

Contents

1 Introduction

The purpose of this paper is to give a detailed and comprehensive account on the existence of marked Gibbs measures in the high temperature and low fugacity regime for general underlying spaces using the method of cluster expansion. Our motivation for this general framework is related to the examples we would like to cover, see Examples 3.3, 3.4, and 3.7 below and also Subsection 5.4. On the other hand in recent papers [AKR98a], [AKR98b], and [Röc98] the authors put special emphasis in the construction of differential geometry on the simple configuration space Γ_X (cf. definition (2.1) below) via a lifting of the geometry from the underlying manifold X. Therefore one aim of this work is to construct marked Gibbs measures for these cases.

The results of this paper (which we will give an account below) are based in the so-called cluster expansion method, see e.g., [MM91], [Rue69], and [Pen63], and we follow closely the ideas of V. A. Malyshev and R. Minlos (cf. [MM91, Chap. 3 and 4]), where these results were announced for the case of $\mathbb{R}^d$. Let us explain this more precisely. Let X be a Riemannian manifold for simplicity, the space describing the position of the particle and S a separable Banach space, the mark space, describing some internal degree of freedom, e.g., spin, momentum, or different types of particles. We construct a marked Poisson measure $\pi^\tau_{z\sigma}$, $z > 0$ over the marked configuration space, i.e.,

$$\Omega := \{\omega = \{(x_1, s_{x_1}), (x_2, s_{x_2}), \ldots\} \in \Gamma_{X\times S} | \{x_1, x_2, \ldots\} =: \gamma_\omega \in \Gamma_X, s_{x_i} \in S\},$$

via Kolmogorov's theorem, see Subsections 2.1 and 2.2 below. The desired measure μ on $\Omega_X(S)$ is obtained as a limit (in a sense to be specified later!) of a family of measures $\Pi^{\sigma^\tau,\phi}_\Lambda$, cf. Subsection 5.1. For finite volume $\Lambda \subset X$ the measure $\Pi^{\sigma^\tau,\phi}_\Lambda$ is defined as a perturbation of the marked Poisson measure π^τ_σ, i.e.,

$$\Pi^{\sigma^\tau,\phi}_\Lambda(\omega, F) := \frac{1\!\!1_{\{Z^{\sigma^\tau,\phi}_\Lambda < \infty\}}(\omega)}{Z^{\sigma^\tau,\phi}_\Lambda(\omega)} \int_\Omega 1\!\!1_F(\omega_{X\setminus\Lambda} \cup \omega'_\Lambda) e^{-E^\phi_\Lambda(\omega_{X\setminus\Lambda}\cup\omega'_\Lambda)} \pi^\tau_\sigma(d\omega'),$$

(cf. Definition 3.1 in Section 3). It is well-known that $\Pi^{\sigma^\tau,\phi}_\Lambda$ is a specification in the sense of [Pre76, Section 6] (see also [Pre79] and [Pre80]) for a given potential ϕ, see Remark 3.2 below for details. As a result the measure μ is absolutely continuous with respect to the marked Poisson measure $\pi^\tau_{z\sigma}$, (cf. Theorem 5.3) and we call these measures marked Gibbs measures. We also proved the DLR equations for μ in case of finite range potentials ϕ, cf. Subsection 5.2, Theorem 5.6.

We would like to emphasize that the above results (specially the one of Theorem 5.3) are strongly related with the procedure of cluster expansion and the estimates obtained there. As usual, this procedure is possible under some conditions on the potential ϕ, namely, stability

$$E^{\phi}(\omega) \geq -B|\omega|, \ \forall \omega \in \Omega_{fin}, \ B \geq 0,$$

and integrability

$$C(\beta) := \operatorname*{ess\,sup}_{(y,s)\in X\times S} \int_X \int_S |e^{-\beta\phi(\hat{x},\hat{y})} - 1|\tau(x,ds)\sigma(dx) < \infty,$$

where $\beta > 0$ is the inverse temperature. The essential moment in the construction is an estimate for (cf. Proposition 4.13 and Appendix A.3)

$$\int_{\Omega_\Lambda \setminus \{\emptyset\}} \int_{\Omega_{fin}} |k(\omega \cup \omega')| \nu_{z\sigma^\tau}(d\omega)\nu_{z\sigma^\tau}(d\omega'),$$

which implies all further convergent results. Here k is the so-called Ursell functions defined by

$$k(\omega) := \sum_{G\in\mathfrak{G}^c(\omega)} \prod_{\{\hat{x},\hat{y}\}\in G} (e^{-\beta\phi(\hat{x},\hat{y})} - 1).$$

As a guideline let us shortely sketch the main steps in this cluster expansion and the proof of its convergence. Using a modifiend function $\bar{k}$ (some kind of "density" of the correlation function) we derive a recursive equation related to the Kirkwood-Salsburg equation, cf. (4.10). the next step is to construct a function Q which dominates $\bar{k}$ and fullfils an easier equation, see (4.14). Due to combinatorical arguments we show that we can reduce the sumation in the bound of Q to the sum over all trees instead of the sum over all connected graphs,

$$|k(\omega)| \leq e^{2\beta B|\omega|} \sum_{T\in\mathfrak{T}(\omega)} \prod_{\{\hat{x},\hat{x}'\}\in T} |e^{\beta\phi(\hat{x},\hat{x}')} - 1|.$$

This opens the door to bound integrals of k and leads to the final estimates.

The convergence of the measures $\Pi_\Lambda^{\sigma^\tau,\phi}$ is after a consequence of the Lebesgue dominated convergence theorem and the cluster decomposition of the Gibbs factor

$$e^{-\beta E^\phi(\omega)} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\omega_1,\dots,\omega_n)\in\mathfrak{P}^n(\omega)} k(\omega_1)\dots k(\omega_n).$$

Thus the contents of Sections 3, 4, and 5 has been described. It remains to add that Section 2 consists of the necessary preliminaries for the further sections. Namely, we give a sketch of the construction of the marked configuration space $\Omega_X(S)$ and its measurable structure, (cf. Subsection 2.1) as well as the marked Poisson measures $\pi_{z\sigma}^\tau$, see Subsection 2.2. The remainder of Section 2 we introduce some algebric structures in order to perform easier calculations and combinatorics involved in cluster expansion. This is the contents of Subsection 2.3 and 2.4. For the clarity of the presentation we moved some proofs to the Appendix A, namely, Lemma 2.9, Proposition 4.11, and 4.13.

Finally we would like to remark that all our results extends to underlying spaces more general than we discuss in the main body of the work, namely, separable standard Borel spaces. The necessary modifications are considered in Subsection 5.3.

2 Marked configurations spaces

In this section we describe the framework to be used on the rest of the paper. Hence in Subsection 2.1 we introduce the measurable structure of the underlying space where the marked Gibbs measure will be defined, see Section 3. Let us mention that such measures are called *states* in the statistical physics of continuous systems and in probability theory they are known as *marked point random fields*, cf. e.g. [AGL78], [GZ93], [Kin93], and [MM91].

The marked Poisson measures are constructed in Subsection 2.2. Finally in Subsection 2.3 and Subsection 2.4 we introduce some facts from graph theory (resp. *-calculus) which will simplify our calculations later on, see Section 4 and also Appendix A.

Let X be a non-compact C^∞ Riemannian manifold (which fulfils the second axiom of countability. i.e., the topology is countably generated). It describes in the usual statistical mechanics the position space of the particle. Denote by $\mathcal{B}(X)$ the Borel σ-algebra on X and by $\mathcal{B}_c(X)$ the set of all elements in $\mathcal{B}(X)$ which have compact closures which are called volumes. Additionally we suppose given a separable Banach space S. The corresponding Borel σ-algebra we denote by $\mathcal{B}(S)$. The elements of this space we call *marks*, they describe an internal degree of freedom.

These assumptions are motivated by the examples given in Section 3, but all considerations the in sequal are still valid under the assumption that X and S are standard Borel spaces, for details see Subsection 5.3.

2.1 The marked configuration space over a manifold

We briefly recall the basic definitions of the simple configuration space over a manifold X for the reader's convenience. The presentation is very much based along the lines of the works by S. Albeverio et al. [AKR98a], but see also [AKR96], [Pre76], [Röc98], and [Shi94].

The *simple configuration space* $\Gamma := \Gamma_X$ over the manifold X is defined as the set of all locally finite subsets (configurations) in X:

$$\Gamma_X := \{\gamma \subset X \,|\, |\gamma \cap K| < \infty \text{ for any compact } K \subset X\}. \tag{2.1}$$

Here (and below) $|A|$ denotes the cardinality of a set A. For any $Y \subset X$ we define

$$\Gamma_Y := \{\gamma \in \Gamma || \gamma \cap (X \backslash Y)| = 0\}.$$

In this paper we are interested in a bigger space of configurations, so-called marked configuration space, thus we proceed giving its abstract definition. For concrete examples see Subsection 3.2.

The *marked configuration space* $\Omega_X(S) := \Omega_X := \Omega$ is defined by

$$\Omega := \{\omega = \{(x_1, s_{x_1}), (x_2, s_{x_2}), \dots\} \in \Gamma_{X\times S} | \{x_1, x_2, \dots\} =: \gamma_\omega \in \Gamma_X, s_{x_i} \in S\}. \tag{2.2}$$

Equivalently Ω can be described as follows

$$\Omega := \{\omega = (\gamma_\omega, s) | \gamma_\omega \in \Gamma_X, s \in S^{\gamma_\omega}\},$$

where S^{γ_ω} stands for the set of all maps $\gamma_\omega \ni x \mapsto s_x \in S$. For any $Y \in \mathcal{B}(X)$ we define in a similar way the space $\Omega_Y(S) := \Omega_Y$. We sometimes use the shorthand ω_Y (resp. γ_Y) for $\omega \cap (Y \times S)$, $Y \subset X$ (resp. $\gamma \cap Y$) and $\hat{x} := (x, s_x) \in X \times S$.

In order to define a measurable structure on Ω we use the following family of sets $\mathfrak{I}$, the "local" sets

$$\mathfrak{I} := \{B \in \mathcal{B}(X) \times \mathcal{B}(S) | \exists \Lambda \in \mathcal{B}_c(X) \text{ with } B \subset \Lambda \times S\}, \tag{2.3}$$

For any $A \in \mathfrak{I}$ define the mapping $N_A : \Omega \to \mathbb{N}_0$ by

$$N_A(\omega) := |\omega \cap A|, \ \omega \in \Omega,$$

then

$$\mathcal{B}(\Omega) := \sigma(\{N_A | A \in \mathfrak{I}\}).$$

For any $Y \in \mathcal{B}(X)$ we define the following σ-algebra on Ω

$$\mathcal{B}_Y(\Omega) := \sigma(\{N_A | A \in \mathfrak{I}, \ A \subset Y \times S\}).$$

For any $Y \in \mathcal{B}(X)$ the σ-algebra $\mathcal{B}_Y(\Omega)$ is isomorphic to $\mathcal{B}(\Omega_Y)$. The "filtration" $(\mathcal{B}_\Lambda(\Omega))_{\Lambda \in \mathcal{B}_c(X)}$ is one of the basic structures in the definition of the Gibbs measures, see Section 5. Moreover if $Y_1, Y_2 \in \mathcal{B}(X)$ such that $Y_1 \cap Y_2 \neq \emptyset$, then $\Omega_{Y_1 \sqcup Y_2}$ is isomorphic to $\Omega_{Y_1} \times \Omega_{Y_2}$.

Finally we want to give another useful description of the marked configuration space Ω. For any $n \in \mathbb{N}_0$ and any $Y \in \mathcal{B}(X)$ we define the n-point configuration space as a subset of Ω_Y by

$$\Omega_Y^{(n)} := \Omega_Y^{(n)}(S) := \{\omega \in \Omega_Y | |\omega| = n\}, \ \Omega_Y^{(0)} := \{\emptyset\},$$

and denote the corresponding σ-algebra by $\mathcal{B}(\Omega_Y^{(n)})$.

There is a bijection

$$\widetilde{(Y \times S)}^n/S_n \to \Omega_Y^{(n)},\ n \in \mathbb{N},\ Y \in \mathcal{B}(X), \tag{2.4}$$

where

$$\widetilde{(Y \times S)}^n := \{((x_1, s_{x_1}), \ldots, (x_n, s_{x_n}))|x_i \in Y, s_{x_i} \in S, x_i \neq x_j, \text{ for } i \neq j\},$$

and S_n is the permutation group over $\{1, \ldots, n\}$. Since this bijection is measurable in both directions the natural σ-algebra on $\widetilde{(Y \times S)}^n/S_n$ is isomorphic to $\mathcal{B}(\Omega_Y^{(n)})$.

One can reconstruct Ω from the sets $\Omega_\Lambda^{(n)}$ using the following scheme. Notice that we can write for any $\Lambda \in \mathcal{B}_c(X)$

$$\Omega_\Lambda = \bigsqcup_{n=0}^{\infty} \Omega_\Lambda^{(n)},$$

hence the σ-algebra $\mathcal{B}(\Omega_\Lambda)$ is the disjoint union of the σ-algebras $\mathcal{B}(\Omega_\Lambda^{(n)})$.

For any $\Lambda_1, \Lambda_2 \in \mathcal{B}_c(X)$ with $\Lambda_1 \subset \Lambda_2$ there are natural maps

$$p_{\Lambda_2,\Lambda_1} : \Omega_{\Lambda_2} \longrightarrow \Omega_{\Lambda_1},$$

$$p_{\Lambda_1} : \Omega \longrightarrow \Omega_{\Lambda_1}$$

defined by $p_{\Lambda_2,\Lambda_1}(\omega) := \omega_{\Lambda_1}$, $\omega \in \Omega_{\Lambda_2}$ (resp. $p_{\Lambda_1}(\omega) = \omega_{\Lambda_1}$, $\omega \in \Omega$). It can be shown that $(\Omega, \mathcal{B}(\Omega))$ coincides with the projective limit of the measurable spaces $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$, $\Lambda \in \mathcal{B}_c(X)$. ???

Finally we would like to introduce one more subspace of Ω which plays a fundamental role in our calculations below, the *finite configuration space* $\Omega_{X,fin} := \Omega_{fin}$. It is defined by

$$\Omega_{fin} := \{\omega \in \Omega|\, |\omega| < \infty\}.$$

The finite configuration space Ω_{fin} has the following useful representation in terms of the n-point configuration spaces

$$\Omega_{fin} = \bigsqcup_{n=0}^{\infty} \Omega_X^{(n)}, \tag{2.5}$$

and in analogous way for $\Omega_{Y,fin}$, $Y \in \mathcal{B}(X)$. The space Ω_{fin} (resp. $\Omega_{Y,fin}$) is equipped with the σ-algebra $\mathcal{B}(\Omega_{fin})$ (resp. $\mathcal{B}(\Omega_{Y,fin})$) of the disjoint unions of measurable spaces $(\Omega_X^{(n)}, \mathcal{B}(\Omega_X^{(n)}))$ (resp. $(\Omega_Y^{(n)}, \mathcal{B}(\Omega_Y^{(n)}))$.

2.2 Marked Poisson measures

For constructing the marked Poisson measure on Ω we need, first of all, to fix an intensity measure σ on the underlying manifold X. Thus, let us assume that σ is a non-atomic Radon measure on X. Additionally, we define a kernel $\tau : X \times \mathcal{B}(S) \to \mathbb{R}$, i.e., $\tau(x, \cdot)$ is a σ-finite measure on $(S, \mathcal{B}(S))$ and $\tau(\cdot, ds)$ $\mathcal{B}(X)$-measurable such that the following condition is fulfilled for any $\Lambda \in \mathcal{B}_c(X)$

$$\int_\Lambda \tau(x, S)\sigma(dx) < \infty. \tag{2.6}$$

This condition will be essential in the estimates later on (cf. proof of Proposition 4.13).

In the product space $X \times S$ we define a measure σ^τ by

$$\sigma^\tau(dx, ds) := \tau(x, ds)\sigma(dx),$$

which is a non-atomic Radon measure.

For any $Y \in \mathcal{B}(X)$ and $n \in \mathbb{N}$ the product measure $\sigma^{\tau \otimes n}$ can be considered as a measure on $(\widetilde{Y \times S})^n$, cf. Lemma A.1 in Appendix A. Let

$$\sigma_n^\tau \restriction \Omega_Y^{(n)} := \sigma^{\tau \otimes n} \circ (\mathrm{sym}_Y^n)^{-1},$$

be the corresponding measure on $\Omega_Y^{(n)}$, where

$$\mathrm{sym}_Y^n : (\widetilde{Y \times S})^n \to \Omega_Y^{(n)},$$

given by

$$\mathrm{sym}_Y^n((\hat{x}_1, \dots, \hat{x}_n)) := \{\hat{x}_1, \dots, \hat{x}_n\} \in \Omega_Y^{(n)}.$$

Then we consider the so-called *Lebesgue-Poisson measure* $\nu_{z\sigma^\tau}$ on $\mathcal{B}(\Omega_{fin})$, which coincides on each $\Omega_X^{(n)}$ with the measure $\sigma_n^\tau \restriction \Omega_X^{(n)}$, as follows

$$\nu_{z\sigma^\tau} := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma_n^\tau \restriction \Omega_X^{(n)}, \tag{2.7}$$

and $\sigma_0^\tau(\emptyset) := 1$. $\nu_{z\sigma^\tau}$ is σ-finite.

Considered as a measure on Ω_Λ, $\Lambda \in \mathcal{B}_c(X)$, the measure $\nu_{z\sigma^\tau}$ is finite with $\nu_{z\sigma^\tau}(\Omega_\Lambda) = e^{z\sigma^\tau(\Lambda \times S)}$. Therefore we define a probability measure $\pi_{z\sigma}^{\tau,\Lambda}$ on Ω_Λ putting

$$\pi_{z\sigma}^{\tau,\Lambda} := e^{-z\sigma^\tau(\Lambda \times S)} \nu_{z\sigma^\tau}.$$

The measure $\pi_{z\sigma}^{\tau,\Lambda}$ has the following property

$$\pi_{z\sigma}^{\tau,\Lambda}(\Omega_\Lambda^{(n)}) = \frac{z^n}{n!}(\sigma^\tau(\Lambda \times S))^n e^{-z\sigma^\tau(\Lambda\times S)},$$

which gives the probability of the occurrence of exactly n points of the marked Poisson process (with arbitrary values of marks) inside the volume Λ.

In order to obtain the existence of a unique probability measure $\pi_{z\sigma}^\tau$ on $(\Omega, \mathcal{B}(\Omega))$ such that

$$\pi_{z\sigma}^{\tau,\Lambda} = \pi_{z\sigma}^\tau \circ p_\Lambda^{-1},\ \Lambda \in \mathcal{B}_c(X),$$

we notice that the family $\{\pi_{z\sigma}^{\tau,\Lambda} | \Lambda \in \mathcal{B}_c(X)\}$ is consistent, i.e.,

$$\pi_{z\sigma}^{\tau,\Lambda_2} \circ p_{\Lambda_2,\Lambda_1}^{-1} = \pi_{z\sigma}^{\tau,\Lambda_1},\ \Lambda_1, \Lambda_2 \in \mathcal{B}_c(X), \Lambda_1 \subset \Lambda_2,$$

and thus, by a version of Kolmogorov's theorem for the projective limit space Ω (cf. [Par67, Chap. V Theorem 5.1] or Theorem 5.12 below) any such family determines uniquely a measure $\pi_{z\sigma}^\tau$ on $\mathcal{B}(\Omega)$ such that $\pi_{z\sigma}^{\tau,\Lambda} = \pi_{z\sigma}^\tau \circ p_\Lambda^{-1}$.

2.3 Basic concepts in graph theory

Now we are going to introduce some standard concepts of graph theory, see e.g. [Ore67].

Let X be a non empty set. A *partition* of X is a collection of non empty subsets $\{X_i\}$ of X such that $X_i \cap X_j = \emptyset$ for $i \neq j$ and $\bigcup_i X_i = X$. The set of all partitions of X where all parts are non-empty is denoted by $\mathfrak{P}(X)$ and by $\mathfrak{P}^n(X)$ we denote the subset of partitions of $\mathfrak{P}(X)$ consisting of n parts. $\mathfrak{P}_\emptyset^n(X)$ stands for the set of all partitions of n parts which might be empty.

We now give the notion of a graph as well as some of its properties. We note here and henceforth that the graphs under consideration are undirected, see [Ore67, Chap. 1] for this notion.

Definition 2.1 *1. A graph $G := G(X)$ is a subset of*

$$\{\{x, y\} \subset X | x \neq y\}.$$

One calls $\{x, y\} \in G$ the edges of the graph and $V(G) := X$ the vertices of the graph. The collection of all such graphs on X is denoted by $\mathfrak{G}(X)$.

2. *Given two graphs* $G_1 \in \mathfrak{G}(X_1)$, $G_2 \in \mathfrak{G}(X_2)$ *with* $G_1 \cap G_2 = \emptyset$, *their sum graph is the graph given by*

$$G_1 \sqcup G_2 = \{\{x, y\} \subset X_1 \cup X_2 | \{x, y\} \in G_1 \sqcup G_2\}.$$

   If G_1 *and* G_2 *have no common vertices, then the sum graph is denoted by* $G_1 \otimes G_2$. *This procedure extends to an arbitrary family* $\{G_i\}$ *of graphs.*

3. *A graph* G *is called connected iff any pair of vertices is connected. The set of all connected graphs in* X *is denoted by* $\mathfrak{G}^c(X)$. *We assume that the single point is a connected graph.*

Proposition 2.2 *(cf. [Ore67, Theorem 2.2.1]) Let* $G \in \mathfrak{G}$ *be given. Then* G *decomposes uniquely into a disjoint sum* $\otimes_i G_i$ *of its connected components.*

Definition 2.3 *A connected graph* G *is called a tree iff it has no loops. The set of all trees on* X *is denoted by* $\mathfrak{T}(X)$.

Proposition 2.4 *(cf. [Ore67, Theorem 4.1.3]) The number of different trees which can be constructed on* n *given vertices is* n^{n-2}.

Remark 2.5 *Since for any* $n \geq 0$ *we have*

$$\sqrt{2\pi n} n^n e^{-n} \leq n! \leq \sqrt{2\pi n} n^n e^{-n} \exp\left(\frac{1}{12(n-1)}\right),$$

it is not hard to see that n^{n-2} *is smaller than* $e^n n!$.

Remark 2.6 *We use the shorhand* $[n]$ *for* $\{1, \ldots, n\}$ *and thus the symbol* $\mathfrak{T}([n])$ *denotes the trees in* $\{1, \ldots, n\}$.

2.4 *-calculus

In this subsection we point out an algebric structure (see, e.g., [MM91] and [Rue69]) which turns out to simplify our notation and calculations later on.

Let $\mathcal{A}$ be the set of all measurable (complex-valued) functions ψ on Ω_{fin}, i.e.,

$$\mathcal{A} := \{\psi : \Omega_{fin} \to \mathbb{C}, \psi \text{ measurable}\}.$$

In $\mathcal{A}$ we introduce the following operation: for any $\psi_1, \psi_2 \in \mathcal{A}$ and $\omega \in \Omega_{fin}$ we define $\psi_1 * \psi_2$ by

$$(\psi_1 * \psi_2)(\omega) := \sum_{(\omega_1,\omega_2)\in\mathfrak{P}^2_\emptyset(\omega)} \psi_1(\omega_1)\psi_2(\omega_2),$$

which is measurable because the restriction to $\Omega^{(n)}$ is of the form

$$(\psi_1 * \psi_2)(\{\hat{x}_1, \ldots, \hat{x}_n\}) = \sum_{(I,J)\in\mathfrak{P}^2_\emptyset([n])} \psi_1(\{\hat{x}_i | i \in I\})\psi_2(\{\hat{x}_j | j \in J\}).$$

The pair $(\mathcal{A}, *)$ together with the natural vector space structure forms a commutative algebra with unit element

$$1^*(\omega) = \begin{cases} 1, & \omega = \emptyset \\ 0, & \omega \neq \emptyset \end{cases} . \tag{2.8}$$

Notice that for any $\psi_1, \ldots \psi_n \in \mathcal{A}$ we have

$$(\psi_1 * \ldots * \psi_n)(\omega) = \sum_{(\omega_1,\ldots,\omega_n)\in\mathfrak{P}^n_\emptyset(\omega)} \psi_1(\omega_1)\ldots\psi_n(\omega_n),\ \omega \in \Omega_{fin}. \tag{2.9}$$

Let us define $\mathcal{A}_+$ as a subset of $\mathcal{A}$ of the form

$$\mathcal{A}_+ := \{\psi \in \mathcal{A} | \psi(\emptyset) = 0\}.$$

For any $\psi \in \mathcal{A}$ and $\psi_+ \in \mathcal{A}_+$ we have

$$\begin{aligned} (\psi * \psi_+)(\emptyset) &= \sum_{(\omega_1,\omega_2)\in\mathfrak{P}^2_\emptyset(\emptyset)} \psi(\omega_1)\psi_+(\omega_2) \\ &= \psi(\emptyset)\psi_+(\emptyset) = 0, \end{aligned}$$

thus it follows that $\mathcal{A}_+$ is an ideal.

We introduce the mapping $\exp^* : \mathcal{A}_+ \to 1^* + \mathcal{A}_+$ by

$$\exp^* \psi := \sum_{n=0}^{\infty} \frac{1}{n!}\psi^{*n} = 1^* + \psi + \frac{1}{2!}\psi^{*2} + \ldots + \frac{1}{n!}\psi^{*n} + \ldots. \tag{2.10}$$

It follows from (2.9) that for any $\psi \in \mathcal{A}_+$

$$\begin{aligned} (\exp^* \psi)(\emptyset) &= 1^*, \\ (\exp^* \psi)(\omega) &= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\omega_1,\ldots,\omega_n)\in\mathfrak{P}^n_\emptyset(\omega)} \psi(\omega_1)\ldots\psi(\omega_n),\ \omega \in \Omega_{fin}\backslash\{\emptyset\}. \end{aligned}$$

Moreover if we define the mapping $\ln^* : 1^* + \mathcal{A}_+ \to \mathcal{A}_+$ by

$$\ln^*(1^* + \psi) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \psi^{*n} = \psi - \frac{1}{2}\psi^{*2} + \ldots + \frac{(-1)^{n-1}}{n}\psi^{*n} + \ldots,$$

then $\exp^*$ and $\ln^*$ are inverse one each other.

Before prove a useful lemma which we will use namely in Section 4 we introduce some notation for simplicity in what folows. $\{\hat{x}\}_1^n := \{\hat{x}_1, \ldots, \hat{x}_n\}$ and $\sigma^\tau(d\hat{x})_1^n := \sigma^\tau(dx_1, ds_{x_1}) \ldots \sigma^\tau(dx_n, ds_{x_n})$.

Lemma 2.7 *Let F, $\psi_1, \ldots, \psi_n$ be measurable functions defined on Ω_{fin}. Then the following equality holds*

$$\begin{aligned} &\int_{\Omega_{fin}} F(\omega)(\psi_1 * \ldots * \psi_n)(\omega)\nu_{z\sigma^\tau}(d\omega) \qquad (2.11) \\ = &\int_{\Omega_{fin}} \cdots \int_{\Omega_{fin}} F(\omega_1 \cup \ldots \cup \omega_n)\psi_1(\omega_1) \ldots \psi_n(\omega_2)\nu_{z\sigma^\tau}(d\omega)_1^n, \end{aligned}$$

whenever all functions positive or one side make sense for the modulus of the functions.

Proof. Let $F, \psi_1, \ldots \psi_p$ be as required above. Then the definition of ν on the right hand side of (2.11) gives

$$\begin{aligned} &\sum_{n_1,\ldots,n_p=0}^{\infty} \frac{z^{n_1+\ldots n_p}}{n_1! \ldots n_p!} \int_{X^{n_1}} \int_{S^{n_1}} \cdots \int_{X^{n_p}} \int_{S^{n_p}} F(\{\hat{x}\}_1^{n_1+\ldots n_p}) \\ &\times \psi_1(\{\hat{x}\}_1^{n_1}) \ldots \psi_p(\{\hat{x}\}_{n_1+\ldots+n_{p-1}+1}^{n_1+\ldots+n_p})\sigma^\tau(d\hat{x})_1^{n_1+\ldots+n_p} \\ = &\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{n_1+\ldots+n_p=n} \frac{n!}{n_1! \ldots n_p!} \int_{X^n} \int_{S^n} F(\{\hat{x}\}_1^n) \\ &\times \psi_1(\{\hat{x}\}_1^{n_1}) \ldots \psi_p(\{\hat{x}\}_{n_1+\ldots+n_{p-1}+1}^{n})\sigma^\tau(dx, ds)_1^n. \end{aligned}$$

Then interchanging the second sum with the integrals and using the definition of $\nu_{z\sigma^\tau}$ we derive the desired result. ■

Corollary 2.8 *For any $\psi \in \mathcal{A}$ such that either $\psi \in L^1(\Omega_{Y,fin}, \nu_{z\sigma^\tau})$, $Y \in \mathcal{B}(X)$ or ψ positive the equality holds*

$$\int_{\Omega_{Y,fin}} (\exp^* \psi)(\omega)\nu_{z\sigma^\tau}(d\omega) = \exp\left(\int_{\Omega_{Y,fin}} \psi(\omega)\nu_{z\sigma^\tau}(d\omega)\right). \qquad (2.12)$$

Lemma 2.9 *Let* $\psi \in \mathcal{A}$ *and* $\Lambda, \Lambda' \in \mathcal{B}_c(X)$ *be given such that* $\Lambda' \subset \Lambda$, *suppose that* $\psi \in L^1(\Omega_{\Lambda\setminus\Lambda'}, \nu_{z\sigma^\tau})$. *Then the following equality holds*

$$\int_{\Omega_{\Lambda\setminus\Lambda'}} (\exp^* \psi)(\omega \cup \omega')\nu_{z\sigma^\tau}(d\omega) \tag{2.13}$$

$$= \exp\left(\int_{\Omega_{\Lambda\setminus\Lambda'}} \psi(\omega)\nu_{z\sigma^\tau}(d\omega)\right) \exp^*\left(\int_{\Omega_{\Lambda\setminus\Lambda'}} 1\!\!1_{\Omega_{fin}\setminus\{\emptyset\}}(\cdot)\psi(\cdot \cup \omega)\nu_{z\sigma^\tau}(d\omega)\right)(\omega'),$$

for $\nu_{z\sigma^\tau}$*-a.e.* $\omega \in \Omega_{\Lambda'}$.

The details of the proof are given in Subsection A.1 of the Appendix A.

Definition 2.10 *For any* $\psi \in \mathcal{A}$ *we define the operator of gradient* D *on* Ω_{fin} *by*

$$(D_{\{\hat{x}\}}\psi)(\omega) := \psi(\omega \cup \{\hat{x}\}),\ \omega \in \Omega_{fin},\ \hat{x} \in X \times S. \tag{2.14}$$

Remark 2.11 *Let us mention that the operator* D *is related with the Poissonian gradient* ∇^P *(see e.g., [AKR98a], [KSSU98], and [NV95]) by* $D_{\{\hat{x}\}} = \nabla^P + \mathbf{1}$, *where* $\mathbf{1}$ *is the identity operator on* $L^2(\Omega, \pi^\tau_{z\sigma}) \otimes L^2(X \times S, \sigma^\tau)$.

Let us state some properties of the operator D which can be easily checked from the definition.

Proposition 2.12

1. $D_{\{\hat{x}\}}D_{\{\hat{y}\}} = D_{\{\hat{y}\}}D_{\{\hat{x}\}}$, *for* $\hat{x}, \hat{y} \in X \times S$.
2. $(D_{\omega'}\psi)(\omega) = \psi(\omega \cup \omega')$, *for* $\omega, \omega' \in \Omega_{fin}$.
3. $D_{\{\hat{x}\}}(\psi_1 * \psi_2)) = (D_{\{\hat{x}\}}\psi_1) * \psi_2 + \psi_1 * (D_{\{\hat{x}\}}\psi_1)$, *for* $\psi_1, \psi_2 \in \mathcal{A}$, $\omega \in \Omega_{fin}$.
4. $D_{\{\hat{x}\}} \exp^* \psi = (\exp^* \psi) * (D_{\{\hat{x}\}}\psi)$, *for* $\psi \in \mathcal{A}$.

3 Marked Gibbs measures

In the last section we introduced the probability measure π_σ^τ on $(\Omega, \mathcal{B}(\Omega))$ (the marked Poisson measure, cf. Subsection 2.2). In this section we will describe a more wide class of probability measures on $(\Omega, \mathcal{B}(\Omega))$, so-called *marked Gibbs measures.*

3.1 Specifications, Gibbs measure, and global conditions

A measurable function $\phi : (X \times S, X \times S) \to \mathbb{R} \cup \{+\infty\}$ is called a *pair potential.* For a given pair potential we define the energy $E^\phi : \Omega_{fin} \to \mathbb{R} \cup \{+\infty\}$ by

$$E^\phi(\omega) := \sum_{\{\hat{x},\hat{y}\}\subset\omega} \phi(\hat{x}, \hat{y}) \tag{3.1}$$

where the sum of the empty set is defined to be zero.

For any $\Lambda \in \mathcal{B}_c(X)$ the *conditional energy* $E_\Lambda^\phi : \Omega \to \mathbb{R}\cup\{+\infty\}$ is defined by

$$E_\Lambda^\Phi(\omega) = E^\phi(\omega_\Lambda) + W(\omega_\Lambda, \omega_{X\setminus\Lambda}),$$

where the term $W(\omega, \omega_{X\setminus\Lambda})$ (the interaction energy between ω and $\omega_{X\setminus\Lambda}$) is given by

$$W(\omega, \omega_{X\setminus\Lambda}) := \begin{cases} \displaystyle\sum_{\hat{x}\in\omega,\hat{y}\in\omega_{X\setminus\Lambda}} \phi(\hat{x}, \hat{y}), & \text{if } \displaystyle\sum_{\hat{x}\in\omega,\hat{y}\in\omega_{X\setminus\Lambda}} |\phi(\hat{x}, \hat{y})| < \infty \\ +\infty, & \text{otherwise} \end{cases} . \tag{3.2}$$

Notice that the energy E^ϕ may be expressed for any $\omega, \omega' \in \Omega_{fin}\setminus\{\emptyset\}$ such that $\gamma_\omega \cup \gamma_{\omega'} = \emptyset$ and as

$$E^\phi(\omega \cup \omega') = E^\phi(\omega) + E^\phi(\omega') + W(\omega, \omega'), \tag{3.3}$$

Now we can define grand canonical Gibbs measures.

Definition 3.1 *For any $\Lambda \in \mathcal{B}_c(X)$ the specification $\Pi_\Lambda^{\sigma^\tau,\phi}$ is defined for any $\omega \in \Omega$, $F \in \mathcal{B}(\Omega)$ by*

$$\begin{aligned} \Pi_\Lambda^{\sigma^\tau,\phi}(\omega, F) \;\; := \;\; & 1\!\!1_{\{\tilde{Z}_\Lambda^{\sigma^\tau,\phi}<\infty\}}(\omega)[\tilde{Z}_\Lambda^{\sigma^\tau,\phi}(\omega)]^{-1} \int_\Omega 1\!\!1_F(\omega_{X\setminus\Lambda} \cup \omega'_\Lambda) \\ & \times \exp[-E_\Lambda^\phi(\omega_{X\setminus\Lambda} \cup \omega'_\Lambda)]\nu_{z\sigma^\tau}(d\omega'), \end{aligned} \tag{3.4}$$

where

$$\tilde{Z}_\Lambda^{\sigma,\phi}(\omega) := \int_\Omega \exp[-E_\Lambda^\phi(\omega_{X\setminus\Lambda} \cup \omega'_\Lambda)]\nu_{z\sigma^\tau}(d\omega'). \tag{3.5}$$

A probability measure μ on $(\Omega, \mathcal{B}(\Omega))$ is called a grand canonical Gibbs measure with interaction potential ϕ iff

$$\mu\Pi_\Lambda^{\sigma^\tau,\phi} = \mu, \text{ for all } \Lambda \in \mathcal{B}_c(X), \tag{3.6}$$

where for any $F \in \mathcal{B}(\Omega)$ the measure $\mu\Pi_\Lambda^{\sigma^\tau,\phi}$ is defined by

$$(\mu\Pi_\Lambda^{\sigma^\tau,\phi})(F) := \int_\Omega d\mu(\omega)\Pi_\Lambda^{\sigma^\tau,\phi}(\omega, F),$$

and (3.6) above are called Dobrushin-Landford-Ruelle (DLR) equations.

Let $\mathcal{G}_{gc}(\sigma^\tau, \phi)$ denote the set of all such probability measures μ.

Remark 3.2 *1. It is well-known that $\{\Pi_\Lambda^{\sigma^\tau,\phi}, \Lambda \in \mathcal{B}_c(X)\}$ is a $\{\mathcal{B}_{X\setminus\Lambda}(\Gamma), \Lambda \in \mathcal{B}_c(X)\}$-specification in the sense of [Pre76, Section 6], see also [Pre79], i.e., for all $\Lambda, \Lambda' \in \mathcal{B}_c(X)$*

(S1) $\Pi_\Lambda^{\sigma^\tau,\phi}(\omega, \Omega) \in \{0, 1\}$ *for all* $\omega \in \Omega$,

(S2) $\Pi_\Lambda^{\sigma^\tau,\phi}(\cdot, Y)$ *is* $\mathcal{B}_{X\setminus\Lambda}(\Omega)$*-measurable for all* $Y \in \mathcal{B}(\Omega)$,

(S3) $\Pi_\Lambda^{\sigma^\tau,\phi}(\cdot, Y \cap Y') = 1\!\!1_{Y'}\Pi_\Lambda^{\sigma^\tau,\phi}(\cdot, Y)$ *for all* $Y \in \mathcal{B}(\Omega)$, $Y' \in \mathcal{B}_{X\setminus\Lambda}(\Omega)$,

(S4) $\Pi_{\Lambda'}^{\sigma^\tau,\phi} = \Pi_{\Lambda'}^{\sigma^\tau,\phi}\Pi_\Lambda^{\sigma^\tau,\phi}$ *if* $\Lambda \subset \Lambda'$*. Here for any* $\omega \in \Omega$, $Y \in \mathcal{B}(\Omega)$

$$(\Pi_{\Lambda'}^{\sigma^\tau,\phi}\Pi_\Lambda^{\sigma^\tau,\phi})(\omega, Y) := \int_\Omega \Pi_\Lambda^{\sigma^\tau,\phi}(\omega', Y)\Pi_{\Lambda'}^{\sigma^\tau,\phi}(\omega, d\omega').$$

2. It can be easily shown that because of (2.7) for all $\Lambda \in \mathcal{B}_c(X)$, $\omega \in \Omega$, $F \in \mathcal{B}(\Omega)$

$$\Pi_\Lambda^{\sigma^\tau,\phi}(\omega, F) = 1\!\!1_{\{\tilde{Z}_\Lambda^{\sigma^\tau,\phi}<\infty\}}(\omega)[\tilde{Z}_\Lambda^{\sigma^\tau,\phi}(\omega)]^{-1}\{1\!\!1_F(\omega_{X\setminus\Lambda})$$
$$+ \sum_{n=1}^\infty \frac{z^n}{n!}\int_{(\Lambda\times S)^n} 1\!\!1_F(\omega_{X\setminus\Lambda} \cup \{\hat{x}\}_1^n)\exp[-E^\phi(\omega_{X\setminus\Lambda} \cup \{\hat{x}\}_1^n)]\sigma^\tau(d\hat{x})_1^n\},$$

where

$$\tilde{Z}_\Lambda^{\sigma^\tau,\phi}(\omega) = 1 + \sum_{n=1}^\infty \frac{z^n}{n!}\int_{(\Lambda\times S)^n} \exp[-E^\phi(\omega_{X\setminus\Lambda} \cup \{\hat{x}\}_1^n)]\sigma^\tau(d\hat{x})_1^n.$$

3. *From properties (S2) and (S3) a probability measure μ on $(\Omega, \mathcal{B}(\Omega))$ is a grand canonical Gibbs measure iff for all $\Lambda \in \mathcal{B}_c(X)$ and all $Y \in \mathcal{B}(X)$ (the set of all elements in $\mathcal{B}(X)$ which are bounded)*

$$\mathbb{E}_\mu[1\!\!1_Y|\mathcal{B}_{X\setminus\Lambda}(\Omega)] = \Pi_\Lambda^{\sigma^\tau,\phi}(\cdot, Y)\ \mu - a.e.,$$

where for a sub-σ-algebra $\Sigma \subset \mathcal{B}(\Omega)$, $\mathbb{E}_\mu[\cdot|\Sigma]$ denotes the conditional expectation with respect to μ given Σ.

We now formulate the conditions on the intearction which will be used in the next section.

(S) (Stability) There exists $B \geq 0$ such that we have

$$E^\phi(\omega) \geq -B|\omega|,\ \forall\omega \in \Omega_{fin}. \tag{3.7}$$

(I) (Integrability) We assume the following integrability condition

$$C(\beta) := \operatorname*{ess\,sup}_{(y,s)\in X\times S} \int_X \int_S |e^{-\beta\phi(\hat{x},\hat{y})} - 1|\tau(x, ds)\sigma(dx) < \infty, \tag{3.8}$$

where $\beta > 0$ is the inverse temperature.

The stability condition (3.8) implies that for every $\omega \in \Omega_{fin}$ there is $\hat{x}_0 \in \omega$ such that

$$\sum_{\hat{x}\in\omega\setminus\{\hat{x}_0\}} \phi(\hat{x}_0, \hat{x}) \geq -2B. \tag{3.9}$$

3.2 Examples

The following examples motivates our construction of marked Gibbs measures which already appeared in the literature. In Subsection 5.4 we will prove that they fullfils the required conditions, namely, stability and integrability.

Example 3.3 *As X we take $\mathbb{R}^d$ with Lebesgue measure and the marked space $S = \mathbb{R}$. The measure τ has support in some compact set from $\mathbb{R}$, i.e.,* $\mathrm{supp}\tau = [-K, K]$ *and $\tau([-K, K]) < \infty$. The potential ϕ is given by*

$$\phi((x, s_x), (y, s_y)) := \Phi(|x - y|) + J(|x - y|)s_x s_y,$$

where the physical potential Φ corresponds to particles with a hard-core r: $\Phi(|x| \leq r) = \infty$ and decreases faster than $|x|^{-(d+\varepsilon)}$, $\varepsilon > 0$, for $|x| \to \infty$, cf. [RZ98, Sect. I]. J has the form

$$J(|x-y|) = \varepsilon |x-y|^{-\rho},\ \varepsilon, > 0,\ \rho > 0. \tag{3.10}$$

??? This model describe the classical gas of rotators (cf. [RZ98]) and the interactions (3.10), has been studied for lattice models by H.-O. Georgii [Geo88], and also used in models for nematic liquid crystals, see e.g., [LR80].

Example 3.4 *Let $L^\theta(\mathbb{R}^d)$ be the Banach space of all continuous loops $S_\theta \to \mathbb{R}^d$ with S_θ the circle of length θ and $\theta = \frac{1}{K_B T}$, K_B Boltzmann, T temperature. On $L^\theta(\mathbb{R}^d)$ we consider the conditional Wiener measure $W_{x,x}(ds_x)dx$ concentrated on the trajectories starting and ending in x, in other words the Brownian bridge measure. The potential is defined by*

$$\phi((x, s_x), (y, s_y)) := \int_0^\theta V(s_x(t) - s_y(t))dt,$$

where $V \in L^1(\mathbb{R}^d)$ and satisfies

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} V(x_i - x_j) \geq -Bn,\ \forall \{x_1, \ldots x_n\} \subset \mathbb{R}^d.$$

Our aim is to handle the loop space as a marked configuration space putting $X = \mathbb{R}^d$. It would be natural to consider as a mark space in the point $x \in X$ the space $L^\theta_x(X)$ of all loops starting and ending in x. In our setting we are forced to put $S = L^\theta(X)$ and the kernel $\tau(x, ds_x) := W_{x,x}(ds_x)$. This implies that the space $L^\theta_x(X)$ has full τ measure. In Subsection 5.4 we will consider this in more details.

This model is a Euclidean version of quantum statistical mechanics for Maxwell-Boltzmann statistics. More explicitly the traces describing the expectation value of operators out of the algebra of observables are represented using Feynman-Kac formula as expectations with respect to the these given Gibbs measures. A beautiful description of this connection for the standard density matrices is given in [Gin71]. Ginibre also consider the cases of the Bose-Einstein and Fermi-Dirac statistics and construct a cluster expansion of the same type as used in this paper (cf. Section 4 below) but with a different concept of localization. We notice that ϕ has not finite range even when V has this property, a more detailed exposition will be the subject of further investigation.

Example 3.5 *(classical gas of rotators ???) Let $\mathbb{T}$ be the one dimentional torus*

$$\phi((x, s_x), (y, s_y)) := \Phi(|x - y|) - J(|x - y|) \cos(\theta_x - \theta_y),$$

where θ_x ???

Example 3.6 *(Phase space ???) Let X be a Riemannian manifold and TX the corresponding tangent boundle of X ... ???*

Example 3.7 *We consider as X the Euclidean space $\mathbb{R}^d$, $d \geq 2$, and the space of marks $S = \{1, \ldots, q\}$, $q \geq 2$. The potential is given by*

$$\phi((x, s_x), (y, s_y)) := \varphi(x - y)(1 - \delta_{s_x, s_y}) + \psi(x - y),$$

where δ_{s_x, s_y} is the Kronecker symbol and $\varphi, \psi : \mathbb{R}^d \to] - \infty, \infty]$ are even measurable functions. We assume that there exist constants $u > 0$ and $0 \leq r_1 \leq r_2 \leq r_3 \leq r_4$ such that

(A1) (strict repulsion of φ) $\varphi \geq 0$ and $\varphi(x) \geq u$ when $|x| \leq r_3$.

(A2) (finite range of φ) $\varphi(x) = 0$ when $|x| \geq r_4$.

(A3) (strong stability and regularity of ψ) either $\psi \geq 0$, or ψ is superstable and lower regular in the sense of [Rue70].

(A4) (short range of repulsion for ψ) $\psi(x) \leq 0$ when $|x| \geq r_2$, and the positive part ψ_+ of ψ satisfies

$$\int_{\{x||x| \geq r_1\}} \psi_+(x) dx < \infty.$$

This model is known as continuum Potts model, cf. [GH96].

Definition 3.8 *For any $m \in \mathbb{N}$ and $\Lambda \in \mathcal{B}_c(X)$ we define the m-point correlation function $\rho_\Lambda^{(m)} : \Omega_\Lambda^{(m)} \to \mathbb{R}$, with empty boundary condition, by*

$$\rho_\Lambda^{(m)}(\{\hat{x}\}_1^m) := \frac{1}{\tilde{Z}_\Lambda^{z\sigma^\tau, \phi}} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{(\Lambda \times S)^n} e^{-\beta E^\phi(\{\hat{x}\}_1^m \cup \{\hat{y}\}_1^n)} \sigma^{\tau \otimes n}(d\hat{y})_1^n.$$

4 Cluster expansion

In this section we derive the cluster expansion of the Gibbs factor $e^{-\beta E^{\phi}(\omega)}$ (cf. 4.7 below) and perform some estimates which leads to the existence of the marked Gibbs measures, see Theorem 5.3.

4.1 Cluster decomposition property

Definition 4.1 *For any $\omega \in \Omega_{fin}$ we define the function k by*

$$k(\omega) := \ln^*(e^{-\beta E^{\phi}})(\omega), \tag{4.1}$$

or equivalently

$$(\exp^* k)(\omega) = e^{-\beta E^{\phi}(\omega)},$$

where E^{ϕ} is defined in (3.1). k is called Ursell function see e.g., [Rue69].

Proposition 4.2 *The partition function $\tilde{Z}_{\Lambda}^{\sigma^{\tau},\phi}(\emptyset)$, $\Lambda \in \mathcal{O}_c(X)$ has the following representation*

$$\tilde{Z}_{\Lambda}^{\sigma^{\tau},\phi}(\emptyset) = \exp\left(\int_{\Omega_{\Lambda}\setminus\{\emptyset\}} k(\omega)\nu_{z\sigma^{\tau}}(d\omega)\right). \tag{4.2}$$

Proof. This result follows from the fact that $e^{-\beta E^{\phi}(\omega)} = (\exp^* k)(\omega)$ and Corollary 2.8. ■

Proposition 4.3 *The Ursell functions k allowed the following representation for any $\omega \in \Omega_{fin}\setminus\{\emptyset\}$*

$$k(\omega) = \sum_{G\in\mathfrak{G}^c(\omega)} \prod_{\{\hat{x},\hat{y}\}\in G} (e^{-\beta\phi(\hat{x},\hat{y})} - 1),$$

and $k(\emptyset) + 0$.

Proof. According to the definition of the energy (cf. 3.1) that

$$\begin{aligned} e^{-\beta E^{\phi}(\omega)} &= \prod_{\{\hat{x},\hat{y}\}\subset\omega} e^{-\beta\phi(\hat{x},\hat{y})} \\ &= \sum_{G\in\mathfrak{G}(\omega)} \prod_{\{\hat{x},\hat{y}\}\in G} (e^{-\beta\phi(\hat{x},\hat{y})} - 1). \end{aligned} \tag{4.3}$$

Recall that every graph G can be decomposed into a direct sum of its connected components, i.e., $G = \bigoplus_{i=1}^{n} G_i$, where G_i is a connected subgraph and $\{V(G_i)\}$ is a partition of ω, (cf. Subsection 2.3). This yields

$$e^{-\beta E^{\phi}(\omega)} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\omega_1,\ldots,\omega_n)\in\mathfrak{P}^n(\omega)} \prod_{l=1}^{n} \sum_{G_l\in\mathfrak{G}^c(\omega_l)} \prod_{\{\hat{x},\hat{y}\}\in G_l} (e^{-\beta\phi(\hat{x},\hat{y})} - 1). \qquad (4.4)$$

Defining $\tilde{k}$ for any $\omega \in \Omega\backslash\{\emptyset\}$ by

$$\tilde{k}(\omega) := \sum_{G\in\mathfrak{G}^c(\omega)} \prod_{\{\hat{x},\hat{y}\}\in G} (e^{-\beta\phi(\hat{x},\hat{y})} - 1), \qquad (4.5)$$

and $\tilde{k}(\emptyset) = 0$. The expression for $e^{-\beta E^{\phi}(\omega)}$ can be writen as

$$e^{-\beta E^{\phi}(\omega)} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\omega_1,\ldots,\omega_n)\in\mathfrak{P}^n(\omega)} \tilde{k}(\omega_1)\ldots\tilde{k}(\omega_n),$$

which coincide with $\exp^* \tilde{k}$. This implies that $\tilde{k} = k$. ∎

Thus the specification can be expressed

$$\Pi_{\Lambda}^{\sigma^\tau,\phi}(\emptyset, \Delta) = \frac{1}{\tilde{Z}_{\Lambda}^{\sigma^\tau,\phi}(\emptyset)} \int_{\Omega_\Lambda} \mathbb{1}_{\Delta}(\omega) \exp^*(k)(\omega) \nu_{z\sigma^\tau}(d\omega),$$

where

$$\tilde{Z}_{\Lambda}^{\sigma^\tau,\phi}(\emptyset) = \int_{\Omega_\Lambda} \exp^*(k)(\omega)\nu_{z\sigma^\tau}(d\omega) = \exp\left(\int_{\Omega_\Lambda\backslash\{\emptyset\}} k(\omega)\nu_{z\sigma^\tau}(d\omega)\right), \qquad (4.6)$$

if $k \in L^1(\Omega_\Lambda, \nu_{z\sigma^\tau})$, cf. Corollary 4.14 below.

Remark 4.4 *The equality*

$$e^{-\beta E^{\phi}(\omega)} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{(\omega_1,\ldots,\omega_n)\in\mathfrak{P}^n(\omega)} k(\omega_1)\ldots k(\omega_n), \qquad (4.7)$$

is known as the cluster decomposition of the Gibbs factor $e^{-\beta E^{\phi}(\omega)}$. We also notice that the function k is $\mathcal{B}(\Omega_{fin})$-measurable.

Next proposition gives a relation between correlations functions (defined in Subsection 3.8) and Ursell functions.

Proposition 4.5 *Let* $\omega \in \Omega_\Lambda^{(m)}$, $\Lambda \in \mathcal{B}_c(X)$ *be given. Define*

$$\bar{k}(\omega, \omega') := (\exp^*(-k) * D_\omega e^{-\beta E^\phi})(\omega'),$$

for $\gamma_\omega \cap \gamma_{\omega'} \neq \emptyset$. *If* $k \in L^1(\Omega_\Lambda, \nu_{z\sigma^\tau})$ *(cf. (4.19)), then*

$$\rho_\Lambda^{(m)}(\omega, \emptyset) = \int_{\Omega_\Lambda} \bar{k}(\omega, \omega') \nu_{z\sigma^\tau}(d\omega').$$

Proof. It follows from the definition of ρ_Λ and (4.6) that

$$\rho_\Lambda^{(m)}(\omega, \emptyset) = \exp\left(-\int_{\Omega_\Lambda} k(\omega')\nu_{z\sigma^\tau}(d\omega')\right) \int_{\Omega_\Lambda} (D_\omega e^{-\beta E^\phi})(\omega')\nu_{z\sigma^\tau}(d\omega').$$

Now taking into account Corollary 2.8 and Lemma 2.7 with $F = 1$ the above equality gives

$$\int_{\Omega_\Lambda} (\exp^*(-k) * D_\omega e^{-\beta E^\phi})(\omega')\nu_{z\sigma^\tau}(d\omega'),$$

thus

$$\bar{k}(\omega, \omega') := (\exp^*(-k) * D_\omega e^{-\beta E^\phi})(\omega').$$

■

Now we derive an explicit relation between k and $\bar{k}$.

Lemma 4.6 *Let* $\omega \in \Omega_{fin} \backslash \{\emptyset\}$ *be given and suppose that* $\hat{x} \in \omega$. *Then* k *and* $\bar{k}$ *are related by the equation*

$$\bar{k}(\{\hat{x}\}, \omega\backslash\{\hat{x}\}) = k(\omega). \tag{4.8}$$

Proof. By definition of $\bar{k}$ and Property 4 of D we have

$$\begin{aligned}
\bar{k}(\{\hat{x}\}, \omega\backslash\{\hat{x}\}) &= (\exp^*(-k) * D_{\{\hat{x}\}} \exp^*(k))(\omega\backslash\{\hat{x}\}) \\
&= (\exp^*(-k) * \exp^*(k) * D_{\{\hat{x}\}} k)(\omega\backslash\{\hat{x}\}) \\
&= (D_{\{\hat{x}\}} k)(\omega\backslash\{\hat{x}\}) \\
&= k(\omega).
\end{aligned}$$

Hence the result is proved. ■

Remark 4.7 *Notice that for any $\omega, \omega' \in \Omega_{fin}\backslash\{\emptyset\}$ with $\gamma_\omega \cap \gamma_{\omega'} = \emptyset$, $\bar{k}$ may be written as*

$$\bar{k}(\omega,\omega') = \sum_{l=1}^{\infty} \frac{1}{l!} \sum_{(\omega_1,\ldots,\omega_l)\in\mathfrak{P}^l(\omega)} \sum_{(\omega_1',\ldots,\omega_l')\in\mathfrak{P}_\emptyset^l(\omega')} k(\omega_1 \cup \omega_1') \ldots k(\omega_l \cup \omega_l'),$$

which is the same as the sum of all graphs where each connected component has at least one vertice in the points of ω, cf. [MM91, Chap. 4].

Remark 4.8 *Our aim now is to find a bound for $\bar{k}$. To this end we derive an equation of recursive type to $\bar{k}$. Let $\omega, \zeta \in \Omega_{fin}$ be such that $\gamma_\omega \cap \gamma_\zeta = \emptyset$ and $\hat{x}_0$ an arbitrary fixed element in ω. To this end we look again into the definition of $\bar{k}$ which can be written as*

$$\bar{k}(\omega,\zeta) = \sum_{\omega'\subset\zeta} (\exp^*(-k))(\zeta\backslash\omega') D_{\{\hat{x}_0\}} e^{-\beta E^\phi(\omega\backslash\{\hat{x}_0\}\cup\omega')}. \tag{4.9}$$

Having in mind the decomposition (3.3) and (3.2) for E^ϕ one obtains

$$\begin{aligned} e^{-\beta W(\{\hat{x}_0\},\omega')} &= \prod_{\hat{x}\in\omega'} e^{-\beta\phi(\hat{x}_0,\hat{x})} \\ &= \sum_{\omega''\subset\omega'} \prod_{\hat{x}\in\omega''} (e^{-\beta\phi(\hat{x}_0,\hat{x})} - 1) \\ &= \sum_{\omega''\subset\omega'} k_{\omega''}(\hat{x}_0), \end{aligned}$$

where

$$k_{\omega''}(\hat{x}_0) := \prod_{\hat{x}\in\omega''} (e^{-\beta\phi(\hat{x}_0,\hat{x})} - 1).$$

According to equation (4.9) now can be formulated as

$$\begin{aligned} \bar{k}(\omega,\zeta) &= e^{-\beta W(\{\hat{x}_0\},\omega\backslash\{\hat{x}_0\})} \sum_{\omega'\subset\zeta} (\exp^*(-k))(\zeta\backslash\omega') \\ &\quad \times \sum_{\omega''\subset\omega'} k_{\omega''}(\hat{x}_0) e^{-\beta E^\phi(\omega\backslash\{\hat{x}_0\}\cup\omega')}. \end{aligned}$$

Interchanging the two sums the right hand side becomes

$$e^{-\beta W(\{\hat{x}_0\},\omega\backslash\{\hat{x}_0\})} \sum_{\omega''\subset\zeta} k_{\omega''}(\hat{x}_0) \sum_{\substack{\omega' \\ \omega''\subset\omega'\subset\xi}} (\exp^*(-k))(\zeta\backslash\omega') e^{-\beta E^\phi(\omega\backslash\{\hat{x}_0\}\cup\omega')},$$

and rewriting the second sum as

$$\begin{aligned}
&\sum_{\breve{\omega}\subset\zeta\backslash\omega''} (\exp^*(-k))(\zeta\backslash(\breve{\omega}\cup\omega''))e^{-\beta E^{\phi}(\omega\backslash\{\hat{x}_0\}\cup\breve{\omega}\cup\omega'')} \\
= &\sum_{\breve{\omega}\subset\zeta\backslash\omega''} (\exp^*(-k))((\zeta\backslash\omega'')\backslash\breve{\omega})(D_{\omega\backslash\{\hat{x}_0\}\cup\omega''}e^{-\beta E^{\phi}})(\breve{\omega}),
\end{aligned}$$

the expression for $\bar{k}$ *can be expressed as*

$$\begin{aligned}
\bar{k}(\omega,\zeta) &= e^{-\beta W(\{\hat{x}_0\},\omega\backslash\{\hat{x}_0\})}\sum_{\omega''\subset\zeta} k_{\omega''}(\hat{x}_0) \\
&\quad\times(\exp^*(-k) * (D_{\omega\backslash\{\hat{x}_0\}\cup\omega''}e^{-\beta E^{\phi}})(\zeta\backslash\omega'').
\end{aligned}$$

Finally taking into account the definition of $\bar{k}$ *we arrive at*

$$\bar{k}(\omega,\zeta) = e^{-\beta W(\{\hat{x}_0\},\omega\backslash\{\hat{x}_0\})}\sum_{\omega''\subset\zeta} k_{\omega''}(\hat{x}_0)\bar{k}(\omega\backslash\{\hat{x}_0\}\cup\omega'',\zeta\backslash\omega''). \tag{4.10}$$

According to the definition of $\bar{k}$ *we have for the case* $\omega = \emptyset$ *that* $\bar{k}(\emptyset,\zeta) = 1^*(\zeta)$, *where* 1^* *is defined by (2.8).*

4.2 Convergence of cluster expansion

We want to derive now a bound for $|\bar{k}(\omega,\zeta)|$ which will be used later on for the main estimation in this section (cf. 4.18). The idea is to define some function Q dominating $\bar{k}$ which fulfills an equation similar to (4.10) and "solve" this equation.

Let us choose a mapping $I : \Omega_{fin} \to X \times \mathbb{S}$, $\tilde{\omega} \mapsto I(\tilde{\omega}) \in \tilde{\omega}$ the following equation is fulfilled

$$\sum_{\hat{x}\in\tilde{\omega}\backslash I(\tilde{\omega})} \phi((\hat{x}), I(\tilde{\omega})) > -2B. \tag{4.11}$$

Such a mapping exists by the stability condition, cf. (3.9).

Of course given I, $\bar{k}$ (4.10) implies

$$\bar{k}(\omega,\zeta) = \exp\left(-\beta\sum_{\hat{x}\in\omega\backslash I(\omega)} \phi(\hat{x}, I(\omega))\right)\sum_{\omega'\subset\zeta} k_{\omega'}(I(\omega))\bar{k}(\omega\backslash I(\omega)\cup\omega',\zeta\backslash\omega'). \tag{4.12}$$

Now we can start defining Q_I inductively. For $\omega = \emptyset$ we define

$$Q(\emptyset, \zeta) := \begin{cases} 1, & \zeta = \emptyset \\ 0, & \zeta \neq \emptyset \end{cases}, \tag{4.13}$$

and by definition of $\bar{k}(\emptyset, \zeta)$ we have $|\bar{k}(\emptyset, \zeta)| \leq Q_I(\emptyset, \zeta)$.

Assume we already have defined Q_I for all $\omega, \zeta \in \Omega_{fin}$, $\omega \neq \emptyset$, $\gamma_\omega \cap \gamma_\zeta = \emptyset$, and $|\omega \cup \zeta| = n$ such that

$$|\bar{k}(\omega, \zeta)| \leq Q_I(\omega, \zeta),$$

is fulfilled. Then if $\omega, \zeta \in \Omega_{fin}$, are such that $\omega \neq \emptyset$, $\gamma_\omega \cap \gamma_\zeta = \emptyset$, and $|\omega \cup \zeta| = n + 1$, we have, applying (4.10) and (4.11)

$$|\bar{k}(\omega, \zeta)| \leq e^{2\beta B} \sum_{\hat{\omega} \subset \zeta} |k_{\hat{\omega}}(I(\omega))| Q_I(\omega \backslash I(\omega) \cup \hat{\omega}, \zeta \backslash \hat{\omega}).$$

Thus we define

$$Q_I(\omega, \zeta) := e^{2\beta B} \sum_{\hat{\omega} \subset \zeta} |k_{\hat{\omega}}(I(\omega))| Q_I(\omega \backslash I(\omega) \cup \hat{\omega}, \zeta \backslash \hat{\omega}), \tag{4.14}$$

and hence we have the following proposition.

Proposition 4.9 *For I, $\bar{k}$ as above there exists a solution Q_I of the equation (4.14) with the initial condition (4.13) which dominates $\bar{k}$, i.e., for any $\omega, \zeta \in \Omega_{fin}$, such that $\gamma_\omega \cap \gamma_\zeta = \emptyset$ we have $|\bar{k}(\omega, \zeta)| \leq Q_I(\omega, \zeta)$.*

Remark 4.10 *The solutions of the equations (4.10) and (4.14) are unique. Let us explain this in more details. On one hand the equations are linear on the other hand the value at the point (ω, ζ) for $|\omega| + |\zeta| = n$ only depends on the values at points $(\tilde{\omega}, \tilde{\zeta})$ with $|\tilde{\omega}| + |\tilde{\zeta}| = n - 1$, thus the corresponding matrix is an strict upper triangle matrix for a suitable choice of the bases.*

The next proposition gives a solution for the equation (4.14) and moreover this solution does not depend on the choice of I.

Proposition 4.11 *The solution of (4.14) and (4.13) for $\omega = \{\hat{x}_1, \ldots, \hat{x}_l\}$, $l \geq 1$ is of the form*

$$Q(\{\hat{x}_1, \ldots, \hat{x}_l\}, \zeta) = \frac{1}{\ln} \sum_{(\zeta_1, \ldots, \zeta_l) \in \mathfrak{P}^l_\emptyset(\zeta)} Q(\{\hat{x}_1\}, \zeta_1) \ldots Q(\{\hat{x}_l\}, \zeta_l), \tag{4.15}$$

where

$$Q(\{\hat{x}\},\zeta) := (e^{2\beta B})^{|\zeta|+1} \sum_{T\in\mathfrak{T}(\{\hat{x}\}\cup\zeta)} \prod_{\{\hat{y},\hat{y}'\}\in T} |e^{\beta\phi(\hat{y},\hat{y}')} - 1|, \tag{4.16}$$

and $Q(\emptyset,\zeta)$ *is defined as in (4.13).*

The proof of this proposition is notacionally quite involved and because of "reordering" of graphs, therefore we give the details in an Appendix (cf. Subsection A.2 below).

As a result we have the following proposition.

Proposition 4.12 *For any* $\omega,\zeta\in\Omega_{fin}$ *such that* $\omega\neq\emptyset$, $\gamma_\omega\cap\gamma_\zeta=\emptyset$, *and* $\omega=\{\hat{x}_1,\ldots,\hat{x}_l\}$, $l\geq 1$ *we have*

$$\begin{aligned} |\bar{k}(\omega,\zeta)| &\leq Q(\{\hat{x}_1,\ldots,\hat{x}_l\},\zeta) \\ &= \sum_{\{\zeta_1,\ldots,\zeta_l\}\in\mathfrak{P}^l_\emptyset(\zeta)} Q_I(\{\hat{x}_1\},\zeta_1)\cdots Q(\{\hat{x}_l\},\zeta_l), \end{aligned}$$

and

$$|k(\omega)| \leq e^{2\beta B|\omega|} \sum_{T\in\mathfrak{T}(\omega)} \prod_{\{\hat{x},\hat{x}'\}\in T} |e^{\beta\phi(\hat{x},\hat{x}')} - 1|. \tag{4.17}$$

Proof. The first part follows from the last proposition and the second part follows from the relation between k and $\bar{k}$ (cf. (4.8)) and (4.16). ■

Proposition 4.13 *Let* $\Lambda\in\mathcal{B}_c(\Omega)$ *be given. Then for any* z *such that* $|z| < \frac{1}{2e}(e^{2\beta B}C(\beta))^{-1}$, *where* $C(\beta)$ *is given by the integrability condition (3.8), we have*

$$\int_{\Omega_\Lambda\setminus\{\emptyset\}} \int_{\Omega_{fin}} |k(\omega\cup\omega')|\nu_{z\sigma^\tau}(d\omega)\nu_{z\sigma^\tau}(d\omega') < \infty. \tag{4.18}$$

We refer to the Appendix A.3 for the proof of this proposition.

As a consequence of the last proposition and Fubini's theorem we have the following corollary.

Corollary 4.14 *For any* $\Lambda\in\mathcal{B}_c(X)$ *we have (notice that* $k(\emptyset)=0$, *see Proposition 4.3)*

$$\int_{\Omega_\Lambda} |k(\omega)|\nu_{z\sigma^\tau}(\omega) < \infty, \tag{4.19}$$

and for ν*-a.a.* $\omega'\in\Omega_{fin}\setminus\{\emptyset\}$

$$\int_{\Omega_{fin}} |k(\omega\cup\omega')|\nu_{z\sigma^\tau}(d\omega) < \infty. \tag{4.20}$$

5 Construction of marked Gibbs measure

5.1 Limiting measures from cluster expansion

Below we construct the marked Gibbs measure on Ω as a limiting measure of the specification $\Pi_\Lambda^{\sigma^\tau,\phi}$ for the empty boundary condition in the weak local sense (cf. Theorem 5.3). We also proved that the resulting limiting measure satisfies the DLR equations if additionally we assume finite range for the potential ϕ, see Theorem 5.6.

Lemma 5.1 *For any $\Lambda, \Lambda' \in \mathcal{B}_c(X)$ such that $\Lambda' \subset \Lambda$ and $F \in \mathcal{B}(\Omega_{\Lambda'})$ the specification $\Pi_\Lambda^{\sigma^\tau,\phi}(\emptyset, p_{\Lambda'}^{-1}(F))$ has the following representation*

$$\begin{aligned}
&\Pi_\Lambda^{\Lambda'}(\emptyset, F) := \Pi_\Lambda^{\sigma^\tau,\phi}(\emptyset, p_{\Lambda'}^{-1}(F)) \\
= \;& \frac{1}{\tilde{Z}_\Lambda^{\Lambda'}(\emptyset)} \int_{\Omega_{\Lambda'}} 1\!\!1_F(\omega) \exp^*\left(1\!\!1_{\Omega_{fin}\backslash\{\emptyset\}}(\cdot) \int_{\Omega_{\Lambda\backslash\Lambda'}} k(\omega' \cup \cdot)\nu_{z\sigma^\tau}(d\omega') \right)(\omega)\nu_{z\sigma^\tau}(d\omega),
\end{aligned}$$

where

$$\tilde{Z}_\Lambda^{\Lambda'}(\emptyset) := \exp\left(\int_{\Omega_{\Lambda'}\backslash\{\emptyset\}} \int_{\Omega_{\Lambda\backslash\Lambda'}} k(\omega \cup \omega')\nu_{z\sigma^\tau}(d\omega')\nu_{z\sigma^\tau}(d\omega) \right).$$

Proof. This is a direct consequence of Lemma 2.9. ■

Proposition 5.2 *Let $\Lambda, \Lambda' \in \mathcal{B}_c(X)$ be such that $\Lambda' \subset \Lambda$.*

1. *Let $k_\Lambda^{\Lambda'}$ be defined by*

$$k_\Lambda^{\Lambda'}(\omega) := \int_{\Omega_{\Lambda\backslash\Lambda'}} 1\!\!1_{\Omega_{fin}\backslash\{\emptyset\}}(\omega) k(\omega \cup \omega')\nu_{z\sigma^\tau}(d\omega'). \tag{5.1}$$

   Then for $\nu_{z\sigma^\tau}$-a.a. $\omega \in \Omega_{fin}\backslash\{\emptyset\}$ we have $\lim_{\Lambda\nearrow X} k_\Lambda^{\Lambda'}(\omega) = k^{\Lambda'}(\omega)$, where

$$k^{\Lambda'}(\omega) = 1\!\!1_{\Omega_{fin}\backslash\{\emptyset\}}(\omega) \int_{\Omega_{X\backslash\Lambda',fin}} k(\omega \cup \omega')\nu_{z\sigma^\tau}(d\omega'). \tag{5.2}$$

2. *We have also that $\lim_{\Lambda\nearrow X} \tilde{Z}_\Lambda^{\Lambda'}(\emptyset) = \tilde{Z}^{\Lambda'}(\emptyset)$, where*

$$\tilde{Z}^{\Lambda'}(\emptyset) = \exp\left(\int_{\Omega_{\Lambda'}\backslash\{\emptyset\}} k^{\Lambda'}(\omega)\nu_{z\sigma^\tau}(d\omega) \right) > 0. \tag{5.3}$$

Proof. 1. We would like to estimate the following quantity

$$\left| k_\Lambda^{\Lambda'}(\omega) - \int_{\Omega_{X\setminus\Lambda',fin}} k(\omega \cup \omega')\nu_{z\sigma^\tau}(d\omega') \right|.$$

According to the definition of $k_\Lambda^{\Lambda'}$ the above quantity is estimated by

$$\int_{\Omega_{X\setminus\Lambda',fin}\setminus\Omega_{\Lambda\setminus\Lambda'}} |k(\omega \cup \omega')|\nu_{z\sigma^\tau}(d\omega'). \tag{5.4}$$

Now let $\{\Lambda_n | n \in \mathbb{N}\}$ be a sequence of increasing volumes such that $\Lambda_n \nearrow X$. Then $\Omega_{\Lambda_n\setminus\Lambda'} \nearrow \Omega_{X\setminus\Lambda',fin}$. On the other hand (4.20) guarantees that there exists a ν-null set $N \in \mathcal{B}(\Omega_{fin})$ such that for all $\omega \in \Omega_{fin}\setminus(N \cup \{\emptyset\})$

$$\int_{\Omega_{fin}} |k(\omega \cup \omega')|\nu_{z\sigma^\tau}(d\omega') < \infty.$$

Therefore by Lebesgue's dominated convergence theorem it follows that (5.4) goes to zero for all fixed $\omega \in \Omega_{fin}\setminus(N \cup \{\emptyset\})$. Hence 1 is proved.

To prove 2 we note that

$$|k_\Lambda^{\Lambda'}(\omega)| \leq \int_{\Omega_{fin}} |k(\omega \cup \omega')|\nu_{z\sigma^\tau}(d\omega'),$$

and thus

$$\int_{\Omega_{\Lambda'}\setminus\{\emptyset\}} |k_\Lambda^{\Lambda'}(\omega)|\nu_{z\sigma^\tau}(d\omega) \leq \int_{\Omega_{\Lambda'}\setminus\{\emptyset\}} \int_{\Omega_{fin}} |k(\omega \cup \omega')|\nu_{z\sigma^\tau}(d\omega')\nu_{z\sigma^\tau}(d\omega) < \infty \tag{5.5}$$

because of (4.18). This implies that

$$\lim_{\Lambda\nearrow X} \int_{\Omega_{\Lambda'}\setminus\{\emptyset\}} k_\Lambda^{\Lambda'}(\omega)\nu_{z\sigma^\tau}(d\omega) = \int_{\Omega_{\Lambda'}\setminus\{\emptyset\}} \int_{\Omega_{X\setminus\Lambda',fin}} k(\omega \cup \omega')\nu_{z\sigma^\tau}(d\omega')\nu_{z\sigma^\tau}(d\omega),$$

and, of course, taking exponential, and having in mind the form of $k^{\Lambda'}$ in (5.2), the desired result (5.3) follows. ■

Theorem 5.3 *The specification $\Pi_\Lambda^{\sigma^\tau,\phi}(\emptyset, d\omega)$ converges in the weak local sense to a measure μ, i.e., for any bounded $\mathcal{B}_{\Lambda'}(\Omega)$-measurable function F we have*

$$\int_\Omega F(\omega)\Pi_\Lambda^{\sigma^\tau,\phi}(\emptyset, d\omega) \to \frac{1}{\tilde{Z}^{\Lambda'}(\emptyset)} \int_{\Omega_\Lambda} F(\omega)(\exp^* k^{\Lambda'})(\omega)\nu_{z\sigma^\tau}(d\omega),$$

and thus

$$\mu^{\Lambda'}(d\omega) = \frac{1}{\tilde{Z}^{\Lambda'}(\emptyset)}(\exp^* k^{\Lambda'})(\omega)\nu_{z\sigma^\tau}(d\omega). \tag{5.6}$$

Proof. Let F be a $\mathcal{B}_{\Lambda'}(\Omega)$, then Lemma 5.1 implies

$$\int_\Omega F(\omega)\Pi_\Lambda^{\sigma^\tau,\phi}(\emptyset, d\omega) = \frac{1}{\tilde{Z}_\Lambda^{\Lambda'}(\emptyset)}\int_{\Omega_{\Lambda'}} f(\omega)\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{(\omega_1,\ldots,\omega_n)\in\mathfrak{P}^n(\omega)}\prod_{i=1}^{n} k_\Lambda^{\Lambda'}(\omega_i)\nu_{z\sigma^\tau}(d\omega),$$

where $F = f \circ p_{\Lambda'}$. According to Proposition 5.2 we know already that $\tilde{Z}_\Lambda^{\Lambda'}(\emptyset)$ converges to $\tilde{Z}^{\Lambda'}(\emptyset)$. In order to use the Lebesgue dominated convergence theorem one should estimate the integrand by a function which is integrable and independent of Λ. An appropriate bound is

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{(\omega_1,\ldots,\omega_n)\in\mathfrak{P}^n(\omega)}\prod_{i=1}^{n}\int_{\Omega_{X\setminus\Lambda',fin}} |k(\zeta\cup\omega_i)|\nu_{z\sigma^\tau}(d\zeta) \\ = \;& \exp^*\left(\mathbb{1}_{\Omega_{fin}\setminus\{\emptyset\}}(\omega)\int_{\Omega_{X\setminus\Lambda',fin}} |k(\zeta\cup\omega)|\nu_{z\sigma^\tau}(d\zeta)\right). \end{aligned}$$

Moreover the integral of the bound is given by

$$\begin{aligned} &\int_{\Omega_{\Lambda'}}\exp^*\left(\mathbb{1}_{\Omega_{fin}\setminus\{\emptyset\}}(\omega)\int_{\Omega_{X\setminus\Lambda',fin}} |k(\zeta\cup\omega)|\nu_{z\sigma^\tau}(d\zeta)\right)\nu_{z\sigma^\tau}(d\omega) \\ = \;& \exp\left(\int_{\Omega_{\Lambda'}\setminus\{\emptyset\}}\int_{\Omega_{X\setminus\Lambda',fin}} |k(\zeta\cup\omega)|\nu_{x\sigma^\tau}(d\zeta)\nu_{z\sigma^\tau}(d\omega)\right) < \infty, \end{aligned}$$

because of Corollary 2.8 and (4.18). Therefore we have the desired result

$$\lim_{\Lambda\nearrow X}\int_{\Omega_{\Lambda'}} F(\omega)\Pi_\Lambda^{\sigma^\tau,\phi}(\emptyset, d\omega) = \int_{\Omega_{\Lambda'}} F(\omega)\frac{1}{Z^{\Lambda'}(\emptyset)}(\exp^* k^{\Lambda'})(\omega)\nu_{z\sigma^\tau}(d\omega).$$

■

The measure from Theorem 5.3 is not concentrated in all Ω, indeed we have the following corollary.

Corollary 5.4 *Let A be a $\mathcal{B}(X\times S)$-measurable set such that $\sigma^\tau(A) = 0$, then the set of marked configurations not touching A, i.e.,*

$$\tilde{\Omega} = \{\omega = (\gamma, s)\in\Omega \,|\, (x, s_x)\in A^c, \forall x\in\gamma\},$$

has full μ-measure.

Proof. Let us prove that $\mu(\tilde{\Omega}^c) = 0$. To this end we write $\tilde{\Omega}^c$ as

$$\begin{aligned}\tilde{\Omega}^c &= \{\omega = (\gamma, s) \in \Omega | (x, s_x) \in A, \text{ for some } x \in \gamma\} \\ &= \bigcup_{n \in \mathbb{N}} p_{\Lambda_n}^{-1}(\{\omega = (\gamma, s) \in \Omega_{\Lambda_n} | (x, s_x) \in A, \text{ for some } x \in \gamma_{\Lambda_n}\}).\end{aligned}$$

Therefore

$$\mu(\tilde{\Omega}^c) \leq \sum_{n=1}^{\infty} \mu^{\Lambda_n}(\{\omega = (\gamma, s) \in \Omega_{\Lambda_n} | (x, s_x) \in A, \text{ for some } x \in \gamma_{\Lambda_n}\}).$$

Since $\mu^{\Lambda_n} \ll \nu_{z\sigma^\tau}$ (cf.(5.6)), it is enough to prove that

$$\nu_{z\sigma^\tau}(\{\omega = (\gamma, s) \in \Omega | (x, s_x) \in A, \text{ for some } x \in \gamma_{\Lambda_n}\}) = 0.$$

According to the definition of $\nu_{z\sigma^\tau}$ (cf. (2.7)) the left hand side of the above equality yields

$$\begin{aligned}&\sum_{m=0}^{\infty} \frac{z^m}{m!} \sigma_m^\tau(\{(\hat{x}_1, \ldots, \hat{x}_m) \in (\Lambda_n \times S)^m / S_m | \hat{x}_i \in A \text{ for some } i\}) \\ \leq &\sum_{m=0}^{\infty} \frac{z^m m}{m!} (\sigma^\tau(\Lambda_n \times S))^{m-1} \sigma^\tau(A),\end{aligned}$$

which is zero since $\sigma^\tau(A) = 0$. ■

Remark 5.5 *Since the projections of tempered Gibbs measures at arbitrary temperature and fugacity are absolutely continuous with respect to the Lebesgue Poisson measure (cf. [Rue69]) the above considerations extends also to all Gibbs measures.*

5.2 Identification with Gibbs measures

If we additionally assume finite range potentials, then the limiting measure from Theorem 5.3 verifies the DLR equations. We state this result in the following Theorem.

Theorem 5.6 *For finite range potential ϕ the measure μ from Theorem 5.3 fulfils the DLR equations.*

Proof. Let $\Lambda \in \mathcal{B}_c(X)$ be given. Then there exists a $\tilde{\Lambda} \in \mathcal{B}_c(X)$ such that $\Lambda \subset \tilde{\Lambda}$ and $\phi((x, s_x), (y, s_y)) = 0$ if $x \in \Lambda$ and $y \in \tilde{\Lambda}^c$. Whence the interaction energy is

$$W(\omega_\Lambda, \omega_{X\setminus\Lambda}) = \sum_{\hat{x}\in\omega_\Lambda} \sum_{\hat{y}\in\omega_{X\setminus\Lambda}} \phi(\hat{x}, \hat{y}) = \sum_{\hat{x}\in\omega_\Lambda} \sum_{\hat{y}\in\omega_{\tilde{\Lambda}\setminus\Lambda}} \phi(\hat{x}, \hat{y}) = W(\omega_\Lambda, \omega_{\tilde{\Lambda}\setminus\Lambda}),$$

and the sums are finite. Thus $\Pi_\Lambda^{\sigma^\tau,\phi}$ may be written as

$$\Pi_\Lambda^{\sigma^\tau,\phi}(\omega, d\zeta) = \frac{1\!\!1_{\{\tilde{Z}_\Lambda^{\sigma^\tau,\phi}<\infty\}}(\omega_{\tilde{\Lambda}\setminus\Lambda})}{\tilde{Z}_\Lambda^{\sigma^\tau,\phi}(\omega_{\tilde{\Lambda}\setminus\Lambda})} e^{-\beta E_\Lambda^\phi(\zeta_\Lambda\cup\omega_{\tilde{\Lambda}\setminus\Lambda})} \nu_{z\sigma^\tau}(d\zeta),$$

which implies that $\Pi_\Lambda^{\sigma^\tau,\phi}(\cdot, F)$ is $\mathcal{B}_{\tilde{\Lambda}}(\Omega)$-measurable for $F \in \mathcal{B}(\Omega)$. Hence for any $F \in \mathcal{B}(\Omega)$, we have for all $\Lambda' \in \mathcal{B}_c(X)$ with $\tilde{\Lambda} \subset \Lambda'$

$$\int_\Omega \Pi_\Lambda^{\sigma^\tau,\phi}(\omega, F)\Pi_{\Lambda'}^{\sigma^\tau,\phi}(\emptyset, d\omega) = \Pi_\Lambda^{\sigma^\tau,\phi}(\emptyset, F),$$

cf. Remark 3.2-(S4). Since $\Pi_\Lambda^{\sigma^\tau,\phi}(\cdot, F)$ is bounded we have

$$\int_\Omega \Pi_\Lambda^{\sigma^\tau,\phi}(\omega, F)\Pi_{\Lambda'}^{\sigma^\tau,\phi}(\emptyset, d\omega) \to \int_\Omega \Pi_\Lambda^{\sigma^\tau,\phi}(\omega, F)\mu(d\omega),$$

when $\Lambda \nearrow X$, because $\Pi_{\Lambda'}^{\sigma^\tau,\phi}(\emptyset, \cdot) \to \mu$ in the weak local sense. Moreover, $\Pi_\Lambda^{\sigma^\tau,\phi}(\emptyset, F) \to \mu(F)$ which implies the DLR equations

$$\int_\Omega \Pi_\Lambda^{\sigma^\tau,\phi}(\omega, F)\mu(d\omega) = \mu(F).$$

■

5.3 Extension to standard Borel spaces

In this subsection we will present a natural generalization of our results. Except in Theorem 5.3 we use nothing else than the measurability structure of X and S and there we apply the theorem of Kolmogorov for ptojective limit. Thus the construction works for X and S separable standard Borel spaces. To this end we recall the definition and properties of separable standard Borel, see e.g., [Coh93], [Geo88] and [Par67].

Definition 5.7 *Let $(X, \mathfrak{F})$ and $(X', \mathfrak{F}')$ be two measurable spaces.*

1. *The spaces $(X, \mathfrak{F})$ and $(X', \mathfrak{F}')$ are called isomorphic iff there exists a measurable bijective mapping $f : X \to X'$ such that its inverse f^{-1} is also measurable.*

2. *$(X, \mathfrak{F})$ and $(X', \mathfrak{F}')$ are called σ-isomorphic iff there exists a bijective mapping $F : \mathfrak{F} \to \mathfrak{F}'$ between the σ-algebras which preserves the operations in a σ-algebra.*

3. *$(X, \mathfrak{F})$ is said to be countable generated iff there exists a denumerable class $\mathfrak{D} \subset \mathfrak{F}$ such that $\mathfrak{D}$ generates $\mathfrak{F}$.*

4. *$(X, \mathfrak{F})$ is said to be separable iff it is countably generated and for each $x \in X$ the set $\{x\} \in \mathfrak{F}$.*

Definition 5.8 *Let $(X, \mathfrak{F})$ be a countably generated measurable space. Then $(X, \mathfrak{F})$ is called standard Borel space iff there exists a Polish space $(X', \mathfrak{F}')$ (i.e., metrizable, complete metric space which fulfills the second axiom of countability and the σ-algebra $\mathfrak{F}'$ coincides with the Borel σ-algebra) such that $(X, \mathfrak{F})$ and $(X', \mathcal{B}(X'))$ are σ-isomorphic.*

Example 5.9

1. *Every locally compact σ-compact space is a standard Borel space.*

2. *Polish spaces are standard Borel spaces.*

We have the following proposition, cf. [Par67, Chap. V, Theorem 2.1].

Proposition 5.10

1. *If $(X, \mathfrak{F})$ is a countable generated measurable space, then there exists $E \subset \{0,1\}^{\mathbb{N}}$ such that $(X, \mathfrak{F})$ is σ-isomorphic to $(E, \mathcal{B}(E))$. Thus $(X, \mathfrak{F})$ is σ-isomorphic to a separable measurable space.*

2. *Let $(X, \mathfrak{F})$ and $(X', \mathfrak{F}')$ be separable measurable spaces. Then $(X, \mathfrak{F})$ is σ-isomorphic to $(X', \mathfrak{F}')$ iff they are isomorphic.*

Finally we state some operations under which separable standard Borel space are closed, see e.g., [Par67] and [Coh93].

Theorem 5.11 *Let $(X_1, \mathfrak{F}_1), (X_2, \mathfrak{F}_2), \ldots$ be separable standard Borel spaces.*

1. *Countable product, sums, and union are separable standard Borel spaces.*

2. *The projective limit is a separable standard Borel space.*

3. *Any measurable subset of a separable standard Borel space is also a separable standard Borel space.*

We need also a version of Kolmogorov's theorem for separable standard Borel spaces.

Theorem 5.12 *(cf. [Par67, Chap. V Theorem 5.1]) Let $(X_\alpha, \mathfrak{F}_\alpha)$, $\alpha \in I$ be separable standard Borel spaces. If $\{\mu_F : F \subseteq I,\ F \text{ finite}\}$ is a consistent family of measures, then there exists a unique measure μ on $\mathfrak{F}^I$ such that $\mu_F(A) = \mu(p_{IF}^{-1}(A))$, for all $A \in \mathfrak{F}^F$ and all finite $F \subseteq I$.*

Let us now apply this general framework to our marked configuration space Ω.

We assume, therefore, that $(X, \mathfrak{X})$, $(S, \mathfrak{S})$ are separable standard Borel spaces.

Since $\mathcal{B}_c(X)$ makes in this generality no sense we have to introduce an abstract concept of "local" sets. Let $\mathfrak{I}_X$ be a subset of $\mathfrak{X}$ with the properties:

(I1) $\Lambda_1 \cup \Lambda_2 \in \mathfrak{I}_X$ for all $\Lambda_1, \Lambda_2 \in \mathfrak{I}_X$.

(I2) If $\Lambda \in \mathfrak{I}_X$ and $A \in \mathfrak{X}$ with $A \subset \Lambda$ then $A \in \mathfrak{I}_X$.

(I3) There exists a sequence $\{\Lambda_n, n \in \mathbb{N}\}$ from $\mathfrak{I}_X$ with $X = \bigcup_{n\in\mathbb{N}} \Lambda_n$ and such that if $\Lambda \in \mathfrak{I}_X$ then $\Lambda \subset \Lambda_n$ for some $n \in \mathbb{N}$.

Then we can construct the marked configuration space as in Subsection 2.1. Our aim is to show that $(\Omega, \mathcal{B}(\Omega)$ is a separable standard Borel space and thus by Theorem 5.12 the measure in Theorem 5.3 exists.

It follows from Theorem 5.11 that for any $\Lambda \in \mathfrak{I}_X$ and any $n \in \mathbb{N}$ the set $(\Lambda \times S)^n$ is a separable standard Borel space. Therefore by the same argument $(\widetilde{\Lambda \times S})^n/S_n$ is also a separable standard Borel space. Now taking into account the isomorphism (cf. 2.4) between $(\widetilde{\Lambda \times S})^n/S_n$ and $\Omega_\Lambda^{(n)}$ the same holds for $\Omega_\Lambda^{(n)}$, see e.g., [Shi94]. Hence Ω_Λ is also a separable standard Borel space as well as $\Omega^{(n)}$ by Theorem 5.11, (1).

Finally, the marked configuration space itself is a separable standard Borel space as the projective limit of the separable standard Borel spaces $(\Omega_\Lambda, \mathcal{B}(\Omega_\Lambda))$, $\Lambda \in \mathfrak{I}_X$.

Furthermore, if on $(X, \mathfrak{X})$ is given a non-atomic measure σ with $\sigma(\Lambda) < \infty$ $\forall \Lambda \in \mathfrak{I}_X$ and a kernel $\tau : X \times \mathfrak{G} \to \mathbb{R}$ which fulfills (2.6), then the procedure from Subsection 2.2 can be done in an analogous way and as a result we obtain a probability measure $\pi^\tau_{z\sigma}$, $z > 0$ on $(\Omega, \mathcal{B}(\Omega))$. Specifications and marked Gibbs measures can also be defined analogously, see e.g., [Pre80]. All the contents of Sections 4 and 5 generalize straightforward and only in Theorem 5.3 we need the assumption that the spaces are standard Borel.

5.4 Examples

Here we will verify that our framework is sufficient to treat all the examples presented in Subsection 3.2. More precisely, each of those examples with potential ϕ defined there produces a Gibbs measure on $(\Omega, \mathcal{B}(\Omega))$. Therefore the main task in this subsection is reduced to verify the stability condition (S) (cf. (3.7)) and integrability condition (I) (cf. (3.8)).

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A Appendix

A.1 Proof of Lemma 2.9

Lemma A.1 *1.* $\sigma^{\tau\otimes n}(\{(\hat{x}_1,\dots,\hat{x}_n)\in(X\times S)^n|\exists i,j\ i\neq j \text{ with } x_i = x_j\}) = 0.$

2. $\sigma^{\tau\otimes n}((X\times S)^n\backslash(\widetilde{X\times S})^n) = 0.$

3. The set $A := \{(\omega,\omega')\in\Omega_{fin}\times\Omega_{fin}|\gamma_\omega\cap\gamma_{\omega'}\neq\emptyset\}$ *has zero* $\nu_{z\sigma^\tau}\otimes\nu_{z\sigma^\tau}$*-measure.*

Proof. 1. Because of the symmetry and the non-atomicity of σ we have

$$\begin{aligned}
&\sigma^{\tau\otimes n}(\{(\hat{x}_1,\dots,\hat{x}_n)\in(X\times S)^n|\exists i,j\ i\neq j \text{ with } x_i = x_j\})\\
=&\binom{n}{2}\sigma^{\tau\otimes n}(\{(\hat{x}_1,\dots,\hat{x}_n)\in(X\times S)^n|x_1=x_2\})\\
=&\binom{n}{2}\sigma^{\tau}(X\times S)^{n-2}\sigma^{\tau\otimes 2}(\{(x,s),(x,t))|x\in X,\ s,t\in S\})\\
=&\ 0
\end{aligned}$$

2. Consequence of 1.
3. By (2.5) we can decompose the set A

$$A = \bigsqcup_{n,m=0}^{\infty}\{(\omega,\omega')\in\Omega^{(n)}\times\Omega^{(m)}|\gamma_\omega\cap\gamma_{\omega'}\neq\emptyset\},$$

then the definition of $\nu_{z\sigma^\tau}$ applied to A yields

$$\nu_{z\sigma^\tau}(A) = \sum_{n,m=0}^{\infty}\frac{z^{n+m}}{n!m!}(\sigma_n^\tau\otimes\sigma_m^\tau)(A_{n,m}),$$

where $A_{n,m}$ is given by

$$\begin{aligned}A_{n,m} := \{(\hat{x}_1,\dots,\hat{x}_n,\hat{y}_1,\dots,\hat{y}_n)\in(\widetilde{X\times S})^n/S_n\times(\widetilde{X\times S})^m/S_m|\\
\exists i,j\ i\neq j \text{ with } x_i = y_j\}.\end{aligned}$$

On the other hand we can estimate $(\sigma_n^\tau\otimes\sigma_m^\tau)(A_{n,m})$ by

$$nm\sigma^{\tau\otimes(n+m)}(\{(\hat{x}_1,\dots,\hat{x}_n,\hat{y}_1,\dots,\hat{y}_n)\in(\widetilde{X\times S})^n\times(\widetilde{X\times S})^m|x_1=y_1\}),$$

then the definition of σ^τ and the non-atomicity of σ implies as above that this last expression is zero. ■

Lemma 2.9 Let $\psi \in \mathcal{A}$ and $\Lambda, \Lambda' \in \mathcal{B}_c(X)$ be given such that $\Lambda' \subset \Lambda$, suppose that $\psi \in L^1(\Omega_{\Lambda\setminus\Lambda'}, \nu_{z\sigma^\tau})$. Then the following equality holds

$$\int_{\Omega_{\Lambda\setminus\Lambda'}} (\exp^* \psi)(\omega \cup \omega')\nu_{z\sigma^\tau}(d\omega) \tag{A.1}$$

$$= \exp\left(\int_{\Omega_{\Lambda\setminus\Lambda'}} \psi(\omega)\nu_{z\sigma^\tau}(d\omega)\right) \exp^*\left(\int_{\Omega_{\Lambda\setminus\Lambda'}} 1\!\!1_{\Omega_{fin}\setminus\{\emptyset\}}(\cdot)\psi(\cdot\cup\omega)\nu_{z\sigma^\tau}(d\omega)\right)(\omega'),$$

for $\nu_{z\sigma^\tau}$-a.e. $\omega \in \Omega_{\Lambda'}$.

Proof. First we clearify the existence of the integrals. It follows from Fubini's theorem that

$$\begin{aligned} &\int_{\Omega_{\Lambda'}}\int_{\Omega_{\Lambda\setminus\Lambda'}} |(\exp^* \psi)(\omega \cup \omega')|\nu_{z\sigma^\tau}(d\omega')\nu_{z\sigma^\tau}(d\omega) \\ \leq &\int_{\Omega_{\Lambda'}}\int_{\Omega_{\Lambda\setminus\Lambda'}} (\exp^* |\psi|)(\omega \cup \omega')\nu_{z\sigma^\tau}(d\omega')\nu_{z\sigma^\tau}(d\omega) \\ = &\exp^*\left(\int_{\Omega_\Lambda} |\psi|(\omega)\nu_{z\sigma^\tau}(d\omega)\right) < \infty, \end{aligned}$$

thus for $\nu_{z\sigma^\tau}$-a.e. $\omega \in \Omega_{\Lambda'}$ $(\exp^* |\psi|)(\cdot \cup \omega')$ and $(\exp^* \psi)(\cdot \cup \omega')$ belongs to $L^1(\Omega_{\Lambda\setminus\Lambda'}, \nu_{z\sigma^\tau})$. Hence the following manipulations are justified for $|\psi|$ and therefore also for the function ψ itself.

The left hand side of (A.1) for $\omega' \neq \emptyset$ is by definition equivalent to

$$\int_{\Omega_{\Lambda\setminus\Lambda'}} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{(\omega_1,\ldots,\omega_n)\in\mathfrak{P}^n_\emptyset(\omega\cup\omega')} \psi(\omega_1)\ldots\psi(\omega_n)\nu_{z\sigma^\tau}(d\omega). \tag{A.2}$$

Without lose of generality we may assume $\gamma_\omega \cap \gamma_{\omega'} = \emptyset$ (cf. Lemma A.1). To each partition $(\omega_1, \ldots, \omega_n) \in \mathfrak{P}^n_\emptyset(\omega \cup \omega')$ we define in one to one form the following objects (we put together the ω_i's which have points from ω')

$$\begin{aligned} J &:= \{i \,|\, \omega_i \subset \omega\} \\ l &:= |J| \\ \eta_i &:= \omega_i,\ \forall i \in J \\ \xi_i &:= \omega_i \cap \omega,\ \forall i \notin J \\ \xi'_i &:= \omega_i \cap \omega',\ \forall i \notin J \\ \eta_0 &:= \omega\setminus(\sqcup_{i\in J}\omega_i), \end{aligned}$$

where $l \in \{0, \dots, n-1\}$; $(\eta_0, \dots, \eta_l) \in \mathfrak{P}_\emptyset^{l+1}(\omega)$; $(\xi_{l+1}, \dots \xi_n) \in \mathfrak{P}_\emptyset^{n-l}(\eta_0)$; $(\xi'_{l+1}, \dots \xi'_n) \in \mathfrak{P}^{n-l}(\omega')$. This implies that (A.2) can be rewritten as

$$\int_{\Omega_{\Lambda \setminus \Lambda'}} \psi(\omega \cup \omega') \nu_{z\sigma^\tau}(d\omega) \tag{A.3}$$
$$+ \sum_{n=2}^{\infty} \sum_{l=0}^{n-1} \frac{1}{l!(n-l)!} \int_{\Omega_{\Lambda \setminus \Lambda'}} \sum_{(\eta_0, \dots, \eta_l) \in \mathfrak{P}_\emptyset^{l+1}(\omega)} \prod_{i=1}^{l} \psi(\eta_i) \varphi_{n,l}(\eta_0) \nu_{z\sigma^\tau}(d\omega),$$

where

$$\varphi_{n,l}(\eta_0) := \sum_{(\xi)_{l+1}^n \in \mathfrak{P}_\emptyset^{n-l}(\eta_0)} \sum_{(\xi')_{l+1}^n \in \mathfrak{P}^{n-l}(\omega')} \prod_{i=l+1}^{n} \psi(\xi'_i \cup \xi_i).$$

Then using Lemma 2.7 we obtain

$$\sum_{n=2}^{\infty} \sum_{l=0}^{n-1} \frac{1}{l!(n-l)!} \left(\int_{\Omega_{Y,fin}} \psi(\omega) \nu_{z\sigma^\tau}(d\omega) \right)^l \int_{\Omega_{Y,fin}} \varphi_{n,l}(\eta_0) \nu_{z\sigma^\tau}(d\eta_0).$$

First we look at the integral of $\varphi_{n,l}$,

$$\int_{\Omega_{\Lambda \setminus \Lambda'}} \sum_{(\xi)_{l+1}^n \in \mathfrak{P}_\emptyset^{n-l}(\eta_0)} \sum_{(\xi')_{l+1}^n \in \mathfrak{P}^{n-l}(\omega')} \prod_{i=l+1}^{n} \psi(\xi'_i \cup \xi_i) \nu_{z\sigma^\tau}(d\eta_0)$$
$$= \sum_{(\xi')_{l+1}^n \in \mathfrak{P}^{n-l}(\omega')} \int_{\Omega_{\Lambda \setminus \Lambda'}} \sum_{(\xi)_{l+1}^n \in \mathfrak{P}_\emptyset^{n-l}(\eta_0)} \prod_{i=l+1}^{n} \psi(\xi'_i \cup \xi_i) \nu_{z\sigma^\tau}(d\eta_0).$$

Once more we apply Lemma 2.7 to the right hand side of the above equality to get

$$\sum_{(\xi)_{l+1}^n \in \mathfrak{P}^{n-l}(\omega')} \prod_{i=l+1}^{n} \int_{\Omega_{\Lambda \setminus \Lambda'}} \psi(\xi'_i \cup \xi_i) \nu_{z\sigma^\tau}(d\xi_i). \tag{A.4}$$

Hence interchanging the sums and putting together (A.3) and (A.4) we get

$$\sum_{l=0}^{\infty} \sum_{n=l+1}^{\infty} \frac{1}{l!} \left(\int_{\Omega_{\Lambda \setminus \Lambda'}} \psi(\omega) \nu_{z\sigma^\tau}(d\omega) \right)^l$$
$$\times \frac{1}{(n-l)!} \sum_{(\xi')_1^{n-l} \in \mathfrak{P}^{n-l}(\omega')} \prod_{i=1}^{n-l} \int_{\Omega_{\Lambda \setminus \Lambda'}} \psi(\omega \cup \omega') \nu_{z\sigma^\tau}(d\omega)$$

$$= \exp\left(\int_{\Omega_{\Lambda\setminus\Lambda'}} \psi(\omega)\nu_{z\sigma^\tau}(d\omega)\right) \exp^*\left(\int_{\Omega_{\Lambda\setminus\Lambda'}} 1\!\!1_{\Omega_{fin}\setminus\{\emptyset\}}(\cdot)\psi(\cdot\cup\omega)\nu_{z\sigma^\tau}(d\omega)\right)(\omega').$$

■

A.2 Proof of Proposition 4.11

Proposition 4.11 The solution of (4.14) for $\omega = \{\hat{x}_1, \dots, \hat{x}_l\}$, $l \geq 1$ has the form

$$Q(\{\hat{x}_1, \dots, \hat{x}_l\}, \zeta) = \frac{1}{l!} \sum_{(\omega_1,\dots,\omega_l)\in\mathfrak{P}^l_\emptyset(\zeta)} Q(\{\hat{x}_1\}, \omega_1) \cdots Q(\{\hat{x}_l\}, \omega_l), \tag{A.5}$$

and $Q(\emptyset, \zeta)$ we define as in (4.13), where

$$Q(\{\hat{x}\}, \zeta) := (e^{2\beta B})^{|\zeta|+1} \sum_{T\in\mathfrak{T}(\{\hat{x}\}\cup\zeta)} \prod_{\{\hat{y},\hat{y}'\}\in T} |e^{\beta\phi(\hat{y},\hat{y}')} - 1|, \tag{A.6}$$

for $\zeta \neq \emptyset$ and $Q(\{\hat{x}\}, \emptyset) := e^{2\beta B}$.

Proof. For $\omega = \emptyset$ it follows by definition, hence we assume $\omega \neq \emptyset$. We prove the result by induction in $|\omega| + |\zeta|$.

For $|\omega| + |\zeta| = 1$ with $\gamma_\omega \cap \gamma_\zeta = \emptyset$ we have $\zeta = \emptyset$ and $\omega = \{\hat{x}\}$. On one hand the r.h.s. of (4.14) yields

$$e^{2\beta B} \sum_{\hat{\omega}\subset\emptyset} Q_I(\hat{\omega}, \emptyset\setminus\hat{\omega}) |k_{\hat{\omega}}(\hat{x})| = e^{2\beta B} Q_I(\emptyset, \emptyset) = e^{2\beta B},$$

on the other hand equation (A.6) gives

$$Q_I(\{\hat{x}\}, \emptyset) = e^{2\beta B} \sum_{T\in\mathfrak{T}(\{\hat{x}\})} \prod_{\{\hat{x},\hat{x}'\}\in\emptyset} |e^{\beta\Phi(\hat{x},\hat{x}')} - 1| = e^{2\beta B}.$$

Thus the initial induction step is verified.

Let us assume that the result is true for $|\omega| + |\zeta| = n - 1$ with $\gamma_\omega \cap \gamma_\zeta = \emptyset$. Choose ω, ζ such that $\omega \neq \emptyset$, $\gamma_\omega \cap \gamma_\zeta = \emptyset$, $|\omega| + |\zeta| = n$, and denote $I(\omega) = x_0 \in \omega$. Using (A.5) for $n - 1$ in the r.h.s. of (4.14) one obtains

$$e^{2\beta B} \sum_{\hat{\omega}\subset\zeta} |k_{\hat{\omega}}(\hat{x}_0)| \sum_{\{\omega'_{\hat{x}}|\hat{x}\in\omega\setminus\{\hat{x}_0\}\cup\hat{\omega}\}\in\mathfrak{P}^{n'}_\emptyset(\zeta\setminus\hat{\omega})} \prod_{\hat{x}\in\omega\setminus\{\hat{x}_0\}} Q(\{\hat{x}\}, \omega'_{\hat{x}}) \prod_{\hat{x}\in\hat{\omega}} Q(\{\hat{x}\}, \omega'_{\hat{x}}), \tag{A.7}$$

where $n' = |\omega| + |\hat{\omega}| - 1$. If $\hat{\omega} \neq \emptyset$, then define $\omega'_{\hat{x}_0} := \hat{\omega} \sqcup \bigsqcup_{\hat{x}\in\hat{\omega}} \omega'_{\hat{x}}$ and we make the following re-arrangement in one to one form

$$\emptyset \neq \hat{\omega} \subset \zeta, \{\omega'_{\hat{x}} | \hat{x} \in \omega\backslash\{\hat{x}_0\} \cup \hat{\omega}\} \in \mathfrak{P}^{|\omega|+|\hat{\omega}|-1}(\zeta\backslash\hat{\omega})$$

$$\updownarrow$$

$$(\{\omega'_{\hat{x}} | \hat{x} \in \omega\}, \hat{\omega}, \{\omega'_{\hat{x}} | \hat{x} \in \hat{\omega}\}),$$

where $\{\omega'_{\hat{x}} | \hat{x} \in \omega\} \in \mathfrak{P}^{|\omega|}_{\emptyset}(\zeta)$, $\omega'_{\hat{x}_0} \neq \emptyset$, $\emptyset \neq \hat{\omega} \subset \omega'_{\hat{x}_0}$, and $\{\omega'_{\hat{x}} | \hat{x} \in \hat{\omega}\} \in \mathfrak{P}^{|\hat{\omega}|}_{\emptyset}(\omega'_{\hat{x}_0}\backslash\hat{\omega})$. With this, the expression in (A.7) can be rewritten as

$$\sum_{\substack{\{\omega'_{\hat{x}}|\hat{x}\in\omega\}\in\mathfrak{P}^{|\omega|}_{\emptyset}(\zeta) \\ \omega'_{\hat{x}_0}\neq\emptyset}} \prod_{\hat{x}\in\omega\backslash\{\hat{x}_0\}} Q(\{\hat{x}\}, \omega'_{\hat{x}}) e^{2\beta B} \sum_{\emptyset\neq\hat{\omega}\subset\omega'_{\hat{x}_0}} |k_{\hat{\omega}}(\hat{x}_0)| \times \sum_{\{\omega'_{\hat{x}}|\hat{x}\in\hat{\omega}\}\in\mathfrak{P}^{|\hat{\omega}|}_{\emptyset}(\omega'_{\hat{x}_0}\backslash\hat{\omega})} \prod_{\hat{x}\in\hat{\omega}} Q(\{\hat{x}\}, \omega'_{\hat{x}}). \tag{A.8}$$

Next we use the explicit form for $Q(\{\hat{x}\}, \omega'_{\hat{x}})$ in (A.6) to write the term

$$e^{2\beta B} \sum_{\emptyset\neq\hat{\omega}\subset\omega'_{\hat{x}_0}} |k_{\hat{\omega}}(\hat{x}_0)| \sum_{\{\omega'_{\hat{x}}|\hat{x}\in\hat{\omega}\}\in\mathfrak{P}^{|\hat{\omega}|}_{\emptyset}(\omega'_{\hat{x}_0}\backslash\hat{\omega})} \prod_{\hat{x}\in\hat{\omega}} Q(\{\hat{x}\}, \omega'_{\hat{x}})$$

as

$$\begin{aligned} &\sum_{\emptyset\neq\hat{\omega}\subset\omega'_{\hat{x}_0}} e^{2\beta B} \sum_{\{\omega'_{\hat{x}}|\hat{x}\in\hat{\omega}\}\in\mathfrak{P}^{|\hat{\omega}|}_{\emptyset}(\omega'_{\hat{x}_0}\backslash\hat{\omega})} \prod_{\hat{x}\in\hat{\omega}} (e^{2\beta B})^{|\omega'_{\hat{x}}|+1} \sum_{T_{\hat{x}}\in\mathfrak{T}(\{\hat{x}\}\cup\omega'_{\hat{x}})} k_{T_{\hat{x}}} |k_{\hat{\omega}}(\hat{x}_0)| \\ = &\sum_{\emptyset\neq\hat{\omega}\subset\omega'_{\hat{x}_0}} (e^{2\beta B})^{1+|\omega'_{\hat{x}_0}\backslash\hat{\omega}|+|\hat{\omega}|} \sum_{\{\omega'_{\hat{x}}|\hat{x}\in\hat{\omega}\}\in\mathfrak{P}^{|\hat{\omega}|}_{\emptyset}(\omega'_{\hat{x}_0}\backslash\hat{\omega})} \prod_{\hat{x}\in\hat{\omega}} \sum_{T_{\hat{x}}\in\mathfrak{T}(\{\hat{x}\}\cup\omega'_{\hat{x}})} k_{T_{\hat{x}}} |k_{\hat{\omega}}(\hat{x}_0)|. \end{aligned} \tag{A.9}$$

We again make the following arrangement: for $\emptyset \neq \hat{\omega} \subset \omega'_{\hat{x}_0}$, $\{\omega'_{\hat{x}} | \hat{x} \in \hat{\omega}\} \in \mathfrak{P}^{|\hat{\omega}|}_{\emptyset}(\omega'_{\hat{x}_0}\backslash\hat{\omega})$, and $\{T_{\hat{x}} | x \in \hat{\omega}\} \in \times_{\hat{x}\in\hat{\omega}} \mathfrak{T}(\{\hat{x}\} \cup \omega'_{\hat{x}})$ we define

$$T := \bigsqcup_{\hat{x}\in\hat{\omega}} T_{\hat{x}} \sqcup \{(\hat{x}, \hat{x}_0) | \hat{x} \in \hat{\omega}\} \in \mathfrak{T}(\omega'_{\hat{x}_0} \cup \{\hat{x}_0\}),$$

and vice versa, given $T \in \mathfrak{T}(\omega'_{\hat{x}_0} \cup \{\hat{x}_0\})$ we define $\hat{\omega}$, $\omega'_{\hat{x}}$, and $\{T_{\hat{x}} | \hat{x} \in \hat{\omega}\}$ by

$$\begin{aligned} \hat{\omega} &:= \{\hat{x} \in V(T) | (\hat{x}, \hat{x}_0) \in T\} \subset \omega'_{\hat{x}_0} \\ T_{\hat{x}_0} \oplus \bigoplus_{\hat{x}\in\hat{\omega}} T_{\hat{x}} &:= T\backslash\{(\hat{x}, \hat{x}_0) | \hat{x} \in \hat{\omega}\} \text{ with } \hat{x} \in V(T_{\hat{x}}) \text{ and } V(T_{\hat{x}_0}) = \hat{x}_0 \\ \omega'_{\hat{x}} &:= V(T_{\hat{x}})\backslash\{\hat{x}\} \end{aligned}$$

Then (A.9) can be written (using A.6) as

$$(e^{2\beta B})^{1+|\omega'_{\hat{x}_0}|} \sum_{T\in\mathfrak{T}(\omega'_{\hat{x}_0}\cup\{\hat{x}_0\})} k_T = Q(\{\hat{x}_0\},\omega'_{\hat{x}_0}).$$

Hence (A.8) now simplifies to

$$\sum_{\substack{\{\omega'_{\hat{x}}|\hat{x}\in\omega\}\in\mathfrak{P}_\emptyset^{|\omega|}(\zeta)\\ \omega'_{\hat{x}_0}\neq\emptyset}} \prod_{\hat{x}\in\omega\setminus\{\hat{x}_0\}} Q(\{\hat{x}\},\omega'_{\hat{x}})Q(\{\hat{x}_0\},\omega'_{\hat{x}_0}).$$

After an explicit calculation for the case $\hat{\omega}\neq\emptyset$ we see that the above expression is nothing but the required form for $Q(\omega,\zeta)$. ■

A.3 Proof of Proposition 4.13

Proposition 4.13 Let $\Lambda\in\mathcal{B}_c(\Omega)$ be given. Then for any z such that $|z|<\frac{1}{2e}(e^{2\beta B}C(\beta))^{-1}$, where $C(\beta)$ is given by the integrability condition (3.8), we have

$$\int_{\Omega_\Lambda\setminus\{\emptyset\}}\int_{\Omega_{fin}} |k(\omega\cup\omega')|\nu_{z\hat{\sigma}}(d\omega)\nu_{z\sigma^\tau}(d\omega')<\infty. \tag{A.10}$$

Proof. The definition of ν and the inequality (4.17) implies the following estimate for (4.18)

$$\sum_{m=1}^{\infty}\sum_{n=0}^{\infty}\frac{(ze^{2\beta B})^{n+m}}{m!n!}\sum_{T\in\mathfrak{T}([m+n])}\int_{\Lambda^m}\int_{X^n}\int_{S^{n+m}}\prod_{(i,,j)\in T}|e^{\beta\Phi(\hat{x}_i,\hat{x}_j)}-1|\sigma^\tau(d\hat{x})_1^{m+n}.$$

We now estimate by induction in $m+n$ the term

$$\int_{\Lambda^m}\int_{X^n}\int_{S^{n+m}}\prod_{(i,,j)\in T}|e^{\beta\phi(\hat{x}_i,\hat{x}_j)}-1|\sigma^\tau(d\hat{x})_1^{m+n}. \tag{A.11}$$

For $m+n=1$ we have $n=0$ and (A.11) reduces to

$$\int_\Lambda\int_S\sigma^\tau(d\hat{x})=\int_\Lambda\tau(x,S)\sigma(dx),$$

which is finite by assumption (2.6).

Let us assume that for all $m+n = N-1$ we have for all $T \in \mathfrak{T}([m+n])$.

$$\int_{\Lambda^m}\int_{X^n}\int_{S^{n+m}} \prod_{(i,j)\in T} |e^{-\beta\Phi(\hat{x}_i,\hat{x}_j)} - 1|\sigma^\tau(d\hat{x})_1^{n+m}$$
$$\leq \int_\Lambda \tau(x,S)\sigma(dx)C(\beta)^{n+m-1}.$$

For the case $n+m = N$ we procceed as follows. Let $T \in \mathfrak{T}([m+n])$ be given. Choose $1+n \leq i_0 \leq m+n$ as a foot point of T such that the integral in $\hat{x}_{i_0}$ is over Λ. Then there exists a final pair $\{j_1, j_2\} \in T$ where $\hat{x}_{j_1}$ is the final vertex and $\hat{x}_{j_1} \neq \hat{x}_{i_0}$. Then we have

$$\int_{\Lambda^m}\int_{X^n}\int_{S^{n+m}} \prod_{(i,j)\in T} |e^{-\beta\phi(\hat{x}_i,\hat{x}_j)} - 1|\sigma^\tau(d\hat{x})_1^{m+n}$$
$$\leq \int_\Lambda\int_S\int_{X^{m+n-1}}\int_{S^{n+m-1}} \prod_{(i,j)\in T} |e^{-\beta\phi(\hat{x}_i,\hat{x}_j)} - 1| \prod_{\substack{l=1\\ l\neq i_0}}^{m+n} \sigma^\tau(d\hat{x}_l)\sigma^\tau(d\hat{x}_{i_0})$$
$$\leq \int_\Lambda\int_S\int_{X^{m+n-2}}\int_{S^{n+m-2}} \prod_{(i,j)\in T\setminus\{j_1,j_2\}} |e^{-\beta\phi(\hat{x}_i,\hat{x}_j)} - 1|$$
$$\times \int_X\int_S |e^{-\beta\phi(\hat{x}_{j_1},\hat{x}_{j_2})} - 1|\sigma^\tau(d\hat{x}_{j_1}) \prod_{\substack{l=1\\ l\neq i_0,j_1}}^{m+n} \sigma^\tau(d\hat{x}_l)\sigma^\tau(d\hat{x}_{i_0})$$
$$\leq C(\beta)\int_\Lambda\int_S\int_{X^{m+n-2}}\int_{S^{n+m-2}} \prod_{(i,j)\in T\setminus\{j_1,j_2\}} |e^{-\beta\phi(\hat{x}_i,\hat{x}_j)} - 1| \prod_{\substack{l=1\\ l\neq i_0,j_1}}^{m+n} \sigma^\tau(d\hat{x}_l)\sigma^\tau(d\hat{x}_{i_0})$$
$$\leq C(\beta)^{n+m-1}\int_\Lambda \tau(x,S)\sigma(dx),$$

where in the last inequality we used the induction step. Thus (4.18) is less or equal than

$$\sum_{m=1}^\infty\sum_{n=0}^\infty \frac{(ze^{2\beta B})^{n+m}}{m!n!}\int_\Lambda \tau(x,S)\sigma(dx)C(\beta)^{n+m-1}\sum_{T\in\mathcal{T}([m+n])} 1. \tag{A.12}$$

It follows from Proposition 2.4 that $|\mathcal{T}([n+m])| = (n+m)^{n+m-2} \leq e^{m+n}(m+n)!$. Hence (A.12) is estimated by

$$\frac{\int_\Lambda \tau(x,S)\sigma(dx)}{C(\beta)}\sum_{m=1}^\infty\sum_{n=0}^\infty \frac{(m+n)!}{m!n!}(zeC(\beta)e^{2\beta B})^{m+n}$$

$$
\begin{aligned}
&= \frac{\int_\Lambda \tau(x,S)\sigma(dx)}{C(\beta)} \sum_{l=0}^{\infty} \left(\sum_{m=1}^{l} \frac{l!}{m!(l-m)!} \right) (zeC(\beta)e^{2\beta B})^l \\
&\leq \frac{\int_\Lambda \tau(x,S)\sigma(dx)}{C(\beta)} \sum_{l=0}^{\infty} (2zeC(\beta)e^{2\beta B})^l,
\end{aligned}
$$

from which the result follows. ■

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