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Probabilistic representation of heat equation of convolution type

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Abstract—In this paper we give a probabilistic representation for the solution of the heat equation of convolution type with a generalized function f as initial condition. The method uses a combination between convolution calculus and the generalized stochastic calculus, namely Itô's formula for generalized functions. Finally, generalization to the stochastic heat equation with a gradient term and generalized coefficients is presented.

1. INTRODUCTION

The purpose of this paper is to give a probabilistic representation of the stochastic heat equation of convolution type. These equations are too singular to be solved in the traditional framework and as a result the solutions are located in an suitable generalized function space. As a tool we use the convolution calculus and the generalized stochastic calculus, namely the Itô formula for generalized functions, cf. Theorem 3.4 below. We are concerned with equations of the following type

$$\frac{\partial}{\partial t}U_t = \frac{1}{2}\Delta U_t + V_t * U_t, \qquad U_0 = f, \tag{1}$$

and

$$\frac{\partial}{\partial t}X_t = \frac{1}{2}\Delta X_t + H_t * \nabla X_t, \qquad X_0 = f, \tag{2}$$

where $t \in [0,T]$ and the coefficients V_t, H_t as well as the initial condition f are generalized functions. The explicit solutions are given in terms of the convolution product $*$ (see for example [OS02], [OS04]) and then we show that they are represented in terms of the expectation of a generalized Brownian functional. More precisely, they have the form

$$U_t = \mathbf{E}^x \left(\tau_{B_t}\left(f * \exp^*\left(\int_0^t V_s ds\right)\right)\right)$$

and

$$X_t = \mathbf{E}^{\tilde{P}^x}\left((\tau_{(0,Y_s)}f) * \exp^*\left(\int_0^t H_s dY_s - \frac{1}{2}\int_0^t H_s^{*2} ds\right)\right),$$

where B_t, $t \in [0,T]$ is a Brownian motion starting at $x \in \mathbb{R}^r$ and Y_t, $t \in [0,T]$ is also Brownian motion with probability law $\tilde{P}^x$ starting at x. Such kind of problems were analyzed by several authors from different point of view under certain conditions on the initial data f and the coefficients V_t and H_t. We would like to mention the work of Rajeev and Thangavelu [RT03] where the authors solved the deterministic heat equation with $V_t = 0$ and f a tempered distribution. Other approaches as in [BDP97], [PVW98], [HØUZ96] and others have strong restriction on the initial condition f, such as continuous and bounded.

2. PRELIMINARIES

2.1. Function spaces $\mathcal{F}_\theta(\mathcal{N}')$ and $\mathcal{G}_\theta(\mathcal{N})$

In this section we will introduce the framework which is necessary later on. We start with a real Hilbert space $\mathcal{H} = L^2(\mathbb{R},\mathbb{R}^d) \otimes \mathbb{R}^r$, $d, r \in \mathbb{N}$ with scalar product $(\cdot,\cdot)$ and norm $|\cdot|$. More precisely, if $(f,x) = ((f_1,\ldots,f_d),(x_1,\ldots,x_r)) \in \mathcal{H}$, then the Hilbertian norm of (f,x) is given by

$$|(f,x)|^2 := \sum_{i=1}^d \int_{\mathbb{R}} f_i^2(u)du + \sum_{i=1}^r x_i^2 = |f|^2_{L^2(\mathbb{R},\mathbb{R}^d)} + |x|^2_{\mathbb{R}^r}.$$

Let us consider the real nuclear triplet

$$\mathcal{M}' = S'(\mathbb{R},\mathbb{R}^d) \otimes \mathbb{R}^r \supset \mathcal{H} \supset S(\mathbb{R},\mathbb{R}^d) \otimes \mathbb{R}^r = \mathcal{M}. \tag{3}$$

The pairing $\langle\cdot,\cdot\rangle$ between $\mathcal{M}'$ and $\mathcal{M}$ is given in terms of the scalar product in $\mathcal{H}$, i.e., $\langle(\omega,x),(\xi,y)\rangle := (\omega,\xi)_{L^2(\mathbb{R},\mathbb{R}^d)} + (x,y)_{\mathbb{R}^r}$, $(\omega,x) \in \mathcal{M}'$ and $(\xi,y) \in \mathcal{M}$. Since $\mathcal{M}$ is a Fréchet nuclear space, then it can be represented as

$$\mathcal{M} = \bigcap_{n=0}^\infty S_n(\mathbb{R},\mathbb{R}^d) \otimes \mathbb{R}^r = \bigcap_{n=0}^\infty \mathcal{M}_n,$$

where $S_n(\mathbb{R},\mathbb{R}^d) \otimes \mathbb{R}^r$ is a Hilbert space with norm square given by $|\cdot|_n^2 + |\cdot|^2_{\mathbb{R}^r}$. We will consider the complexification of the triple (3) and denote it by

$$\mathcal{N}' \supset \mathcal{Z} \supset \mathcal{N}, \tag{4}$$

where $\mathcal{N} = \mathcal{M} + i\mathcal{M}$ and $\mathcal{Z} = \mathcal{H} + i\mathcal{H}$. On $\mathcal{M}'$ we have the standard Gaussian measure μ given by Minlos' theorem via its characteristic functional for every $(\xi,p) \in \mathcal{M}$ by

$$C_\mu(\xi,p) = \int_{\mathcal{M}'} \exp(i\langle(\omega,x),(\xi,p)\rangle)d\mu((\omega,x)) = \exp(-\frac{1}{2}(|\xi|^2 + |p|^2)).$$

In order to solve the Cauchy problem (1) and (2) we need to introduce an appropriate space of generalized functions for which we follow closely the construction in [JOO02].

Let $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be a continuous, convex, increasing function satisfying

$$\lim_{t\to\infty} \frac{\varphi(t)}{t} = \infty \qquad \text{and} \quad \varphi(0) = 0.$$

Such a function is called a Young function. For a Young function φ we define

$$\varphi^*(x) := \sup_{t\geq 0}\{tx - \varphi(t)\}.$$

This is called the polar function associated to φ. It is known that φ^* is again a Young function and $(\varphi^*)^* = \varphi$, see [KR61] for more details and general results.

Given a Banach space B, we denote by $H(B)$ the space pf all entire functions on B, i.e., of all continuous functions $f : B \longrightarrow \mathbb{C}$ such that for every $x, y \in B$ the map

$$\mathbb{C} \ni z \mapsto f(x + zy) \in \mathbb{C}$$

is an entire function on $\mathbb{C}$.

Let $\theta = (\theta_1, \theta_2)$ be a fixed pair of Young functions and $m = (m_1, m_2)$ a given pair of strictly positive real numbers ($m \in (\mathbb{R}_+^*)^2$ for short). We define the Banach space $\mathcal{F}_{\theta,m}(\mathcal{N}_{-n})$, $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$ by

$$\mathcal{F}_{\theta,m}(\mathcal{N}_{-n}) := \{f \in H(\mathcal{N}_{-n});\ |f|_{\theta,m,n} < \infty\},$$

where

$$|f|_{\theta,m,n} := \sup_{(\omega,x)\in\mathcal{N}_{-n}} |f(z)| \exp(-\theta_1(m_1|\omega|_{-n}) - \theta_2(m_2|x|)).$$

The Hilbert norm $|\omega|_{-n}$ is the norm in the dual space $S_n'(\mathbb{R}, \mathbb{R}^d) =: S_{-n}(\mathbb{R}; \mathbb{R}^d)$. For short in the following we denote elements of $\mathcal{N}_{-n}$ by z and $\theta(m|z|_{-n}) := \theta_1(m_1|\omega|_{-n}) + \theta_2(m_2|x|)$.

Then $\{\mathcal{F}_{\theta,m}(\mathcal{N}_{-n}),\ m \in (\mathbb{R}_+^*)^2,\ n \in \mathbb{N}_0\}$ becomes a projective system of Banach spaces. We consider as test function space

$$\mathcal{F}_\theta(\mathcal{N}') := \bigcap_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0} \mathcal{F}_{\theta,m}(\mathcal{N}_{-n})$$

endowed with the projective limit topology. $\mathcal{F}_\theta(\mathcal{N}')$ is called the space of entire functions on $\mathcal{N}'$ of θ-exponential growth of minimal type.

On the other hand, $\{\mathcal{F}_{\theta,m}(\mathcal{N}_n),\ m \in (\mathbb{R}_+^*)^2,\ n \in \mathbb{N}_0\}$ becomes an inductive system of Banach spaces. Then the space of entire functions on $\mathcal{N}$ with θ-exponential growth of finite type is defined by

$$\mathcal{G}_\theta(\mathcal{N}) := \bigcup_{m\in(\mathbb{R}_+^*)^2, n\in\mathbb{N}_0} \mathcal{F}_{\theta,m}(\mathcal{N}_n)$$

endowed with the inductive limit topology.

Remark 2.1 By definition a function $f \in \mathcal{F}_\theta(\mathcal{N}')$ and $\phi \in \mathcal{G}_\theta(\mathcal{N})$ admit the Taylor expansions

$$\begin{aligned} f(z) &= \sum_{k\in\mathbb{N}_0^2} \langle z^{\otimes k}, f_k\rangle, \quad z \in \mathcal{N}',\ f_k = \frac{1}{k!} f^{(k)}(0) \in \mathcal{N}^{\hat{\otimes} k} \qquad (5)\\ g(u) &= \sum_{k\in\mathbb{N}_0^2} \langle u^{\otimes k}, g_k\rangle, \quad u \in \mathcal{N},\ g_k \in \mathcal{N}'^{\hat{\otimes} k}. \end{aligned}$$

The Taylor series map $\mathcal{T}$ (at zero) associates to any entire function the sequence of coefficients. For example, the Taylor series map of f given in (5) is defined by $\mathcal{T}f = \vec{f} = (f_k)_{k\in\mathbb{N}_0^2}$.

2.2. Weighted Fock spaces $F_{\theta,m}(\mathcal{N}_p)$ and $G_{\theta,m}(\mathcal{N}_{-p})$

In order to characterize $\mathcal{F}_\theta(\mathcal{N}')$ and $\mathcal{G}_\theta(\mathcal{N})$ in terms of the Taylor expansions, we introduce weighted Fock spaces. Suppose a pair $k=(k_1,k_2)\in\mathbb{N}_0^2$ and $n\in\mathbb{N}_0$ are given. First we define

$$\theta_{i,k_i} := \inf_{t>0}\frac{e^{\theta_i(t)}}{t^{k_i}}, \qquad i=1,2.$$

For $\vec{f}=(f_k)_{k\in\mathbb{N}_0^2}$ with $f_k\in\mathcal{N}_n^{\hat{\otimes}k}$ we put

$$|\vec{f}|^2_{F_{\theta,m}(\mathcal{N}_n)} := \sum_{k\in\mathbb{N}_0^2}\theta_k^{-2}m^{-k}|f_k|_n^2$$

where $\theta_k^{-2} := \theta_{1,k_1}^{-2}\theta_{2,k_2}^{-2}$. Define

$$F_{\theta,m}(\mathcal{N}_n) := \{\vec{f}=(f_k)_{k\in\mathbb{N}_0},\ f_k\in\mathcal{N}_n^{\hat{\otimes}k};\ |\vec{f}|^2_{F_{\theta,m}(\mathcal{N}_n)}<\infty\}$$

and

$$F_\theta(\mathcal{N}) = \bigcap_{m\in(\mathbb{R}_+^*)^2,n\in\mathbb{N}_0} F_{\theta,m}(\mathcal{N}_n)$$

endowed with the projective limit topology. In the case where $\theta(t)=t^2$, then $F_{\theta,1}(\mathcal{N}_n)$ is nothing than the usual Bosonic Fock space associated to $\mathcal{N}_n$, see [RS75] for more details.

Theorem 2.2 (cf. [JOO02, Thm 2]) *The Taylor series map*

$$\mathcal{T}:\mathcal{F}_\theta(\mathcal{N}')\longrightarrow F_\theta(\mathcal{N}),\ f\mapsto\vec{f}=(f_k)_{k\in\mathbb{N}_0^2}$$

is a topological isomorphism.

Remark 2.3 The space $F_\theta(\mathcal{N})$ is a nuclear Fréchet space which is reflexive and thus $\mathcal{F}_\theta(\mathcal{N}')$ is also a nuclear Fréchet reflexive space.

Similarly, for any $m\in(\mathbb{R}_+^*)^2$ and $n\in\mathbb{N}_0$ we define the Hilbert space

$$G_{\theta,m}(\mathcal{N}_{-n}) = \{\vec{\phi}=(\phi_k)_{k\in\mathbb{N}_0^2};\ |\vec{\phi}|^2_{G_{\theta,m}(\mathcal{N}_{-n})}<\infty\},$$

where

$$|\vec{\phi}|^2_{G_{\theta,m}(\mathcal{N}_{-n})} := \sum_{k\in\mathbb{N}_0^2}(k!\theta_k)^2m^k|\phi_k|^2_{-n}.$$

Then $\{G_{\theta,m}(\mathcal{N}_{-n}),\ m\in(\mathbb{R}_+^*)^2,\ n\in\mathbb{N}_0\}$ becomes an inductive system of Hilbert spaces and we put

$$G_\theta(\mathcal{N}') = \bigcup_{m\in(\mathbb{R}_+^*)^2,n\in\mathbb{N}_0} G_{\theta,m}(\mathcal{N}_{-n})$$

endowed with the inductive limit topology.

By general duality theory the topological dual of $F_\theta(\mathcal{N})$ is identified with the space $G_\theta(\mathcal{N}')$ with the dual pairing:

$$\langle\!\langle\vec{\phi},\vec{f}\rangle\!\rangle = \sum_{k\in\mathbb{N}_0^2}k!\langle\phi_k,f_k\rangle, \qquad \vec{\phi}\in G_\theta(\mathcal{N}'),\ \vec{f}\in F_\theta(\mathcal{N}).$$

Finally, we obtain the following:

Theorem 2.4 (cf. [JOO02, Thm 4]) *The Taylor series map $\mathcal{T}$ yields a topological isomorphism*

$$\mathcal{T} : \mathcal{G}_{\theta^*}(\mathcal{N}) \longrightarrow G_\theta(\mathcal{N}'),\ \phi \mapsto \vec{\phi} = (\phi_k)_{k \in \mathbb{N}_0^2}.$$

We would like to construct the triplet of the complex Hilbert space $L^2(\mathcal{M}', \mu)$ by $\mathcal{F}_\theta(\mathcal{N}')$. To this end we need to add a condition on the pair of Young functions $\theta = (\theta_1, \theta_2)$. Namely,

$$L := \limsup_{t \to \infty} \frac{\theta_i(t)}{t^2} < \infty, \quad i = 1, 2.$$

This is enough to obtain the following Gelfand triplet

$$\mathcal{F}'_\theta(\mathcal{N}') \supset L^2(\mathcal{M}', \mu) \supset \mathcal{F}_\theta(\mathcal{N}'), \tag{6}$$

where $\mathcal{F}'_\theta(\mathcal{N}')$ is the topological dual of $\mathcal{F}_\theta(\mathcal{N}')$ with respect to $L^2(\mathcal{M}', \mu)$ endowed with the inductive limit topology which coincides with the strong topology since $\mathcal{F}_\theta(\mathcal{N}')$ is a nuclear space, see [GV68] for more details on this subject. We denote the duality between $\mathcal{F}'_\theta(\mathcal{N}')$ and $\mathcal{F}_\theta(\mathcal{N}')$ by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ which is the extension of the inner product in $L^2(\mathcal{M}', \mu)$.

The goal is to characterize the generalized functions from $\mathcal{F}'_\theta(\mathcal{N}')$. This will be done in Theorem 2.5 with the help of the Laplace transform. Therefore, let us first define the Laplace transform of an element in $\mathcal{F}'_\theta(\mathcal{N}')$. For every fixed element $(\xi, p) \in \mathcal{N}$ we define the exponential function $\exp((\xi, p))$ by

$$\mathcal{N}' \ni (\omega, x) \mapsto \exp(\langle \omega, \xi \rangle + (p, x)). \tag{7}$$

It easy to verify that for every element $(\xi, p) \in \mathcal{N}$ $\exp((\xi, p)) \in \mathcal{F}_\theta(\mathcal{N}')$. Then for every $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ the Laplace transform of Φ is defined by

$$\hat{\Phi}(\xi, p) := (\mathcal{L}\Phi)(\xi, p) := \langle\!\langle \Phi, \exp((\xi, p)) \rangle\!\rangle. \tag{8}$$

Since the dual of $F_\theta(\mathcal{N})$ is $G_\theta(\mathcal{N}')$, we deduce from Theorem 2.2 and Theorem 2.4 the following diagram

$$\begin{array}{ccc} \mathcal{F}'_\theta(\mathcal{N}') & \xrightarrow{\ \mathcal{L}\ } & \mathcal{G}_{\theta^*}(\mathcal{N}) \\ {\scriptstyle (\mathcal{T}^*)^{-1}} \downarrow & & \downarrow {\scriptstyle \mathcal{T}} \\ F'_\theta(\mathcal{N}) & \longrightarrow & G_\theta(\mathcal{N}') \end{array}$$

Hence, for any $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ we have

$$\begin{aligned} (\mathcal{L}\Phi)(\xi, p) &= \langle\!\langle \Phi, \exp((\xi, p)) \rangle\!\rangle \\ &= \langle\!\langle \vec{\Phi}, \vec{\exp}((\xi, p)) \rangle\!\rangle \\ &= \sum_{k \in \mathbb{N}_0^2} k! \left\langle \vec{\Phi}, \frac{(\xi, p)^{\otimes k!}}{k!} \right\rangle \\ &= (\mathcal{T}\vec{\Phi})(\xi, p). \end{aligned}$$

The action of a generalized function $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ on a test function $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ is given by

$$\langle\!\langle \Phi, \varphi \rangle\!\rangle = \langle\!\langle \vec{\Phi}, \vec{\varphi} \rangle\!\rangle,$$

where $\vec{\Phi} = (\mathcal{T}^*)^{-1}\Phi$ and $\vec{\varphi} := \mathcal{T}\varphi$, [JOO02] for more details.

Thus we have the following characterization theorem:

Theorem 2.5 *The Laplace transform is a topological isomorphism:*

$$\mathcal{L} : \mathcal{F}'_\theta(\mathcal{N}') \longrightarrow \mathcal{G}_{\theta^*}(\mathcal{N}).$$

In the white noise analysis framework Theorem 2.5 is known as Potthoff-Streit characterization theorem, see [PS91] [KLP+96] for details and historical remarks.

Remark 2.6 In Section 3 we will define the stochastic integral of a $\mathcal{F}'_\theta(\mathcal{N}')$-valued process. We shall use the theory of stochastic integration on Hilbert spaces developed in [Mét82]. We have in mind the following considerations. For any $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ there exists $m \in (\mathbb{R}^*_+)^2$ and $n \in \mathbb{N}_0$ such that $\mathcal{T} \circ \mathcal{L}\Phi$ belongs to the Hilbert space $G_{\theta,m}(\mathcal{N}_{-n})$.

2.3. The Convolution Product $*$

Now let us define the convolution between a generalized and a test function from $\mathcal{F}'_\theta(\mathcal{N}')$ and $\mathcal{F}_\theta(\mathcal{N}')$, respectively. Let $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ be given, then the convolution $\Phi * \varphi$ is defined by

$$(\Phi * \varphi)(\omega, x) := \langle\langle \Phi, \tau_{-(\omega,x)}\varphi \rangle\rangle,$$

with $\tau_{-(\omega,x)}\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ being the translation operator of φ, i.e.,

$$(\tau_{-(\omega,x)}\varphi)(\eta, y) := \varphi(\omega + \eta, x + y).$$

It can be proved that $\Phi * \varphi$ is an element of $\mathcal{F}_\theta(\mathcal{N}')$, cf. [GHKO00, Proposition 2.3]. Note that the dual pairing between $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ is given in terms of the convolution product of Φ and φ applied at $(0,0)$, i.e., $\langle\langle \Phi, \varphi \rangle\rangle = (\Phi * \varphi)(0,0)$.

We can generalize the above convolution product for generalized functions as follows. Let $\Phi, \Psi \in \mathcal{F}'_\theta(\mathcal{N}')$ be given. Then $\Phi * \Psi$ is defined as

$$\langle\langle \Phi * \Psi, \varphi \rangle\rangle := \langle\langle \Phi, \Psi * \varphi \rangle\rangle, \ \forall \varphi \in \mathcal{F}_\theta(\mathcal{N}'). \tag{9}$$

In particular, for $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ and $(\xi, p) \in \mathcal{N}$, we have

$$\Phi * \exp((\xi, p)) = (\mathcal{L}\Phi)(\xi, p) \exp((\xi, p)).$$

We then have

$$\mathcal{L}(\Phi * \Psi) = \mathcal{L}\Phi\mathcal{L}\Psi. \tag{10}$$

For every generalized function $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ we define $\exp^* \Phi$, the convolution exponential functional of Φ, by

$$\mathcal{L}(\exp^* \Phi) = \exp(\mathcal{L}\Phi),$$

which is an element of the space $\mathcal{F}'_{(e^{\theta^*})^*}(\mathcal{N}')$.

Remark 2.7 We remark that the Laplace transform may be written as follows

$$\begin{aligned}(\mathcal{L}\Phi)(\xi, p) &= \langle\langle \Phi, \exp((\xi, p)) \rangle\rangle \\ &= \langle\langle \Phi, \exp(\xi) \otimes \exp(p) \rangle\rangle \\ &= (\mathcal{L}_1(\mathcal{L}_2\Phi)(p))(\xi),\end{aligned}$$

where $\mathcal{L}_1$ is the Laplace transform with respect to the first variable and $\mathcal{L}_2$ with respect to the second one. In particular, if $\Phi = \Phi_1 \otimes \Phi_2$ is a generalized function, with $\Phi_1 \in \mathcal{F}'_{\theta_1}(S'_{d.\mathbb{C}})$ ($S'_{d,\mathbb{C}}$ is the complexification of $S'(\mathbb{R},\mathbb{R}^d)$) and $\Phi_2 \in \mathcal{F}'_{\theta_2}(\mathbb{C}^r)$, then we have

$$\begin{aligned}(\mathcal{L}\Phi_1 \otimes \Phi_2)(\xi,p) &= \langle\langle \Phi_1 \otimes \Phi_2, \exp((\xi,p))\rangle\rangle \\ &= \langle\langle \Phi_1 \otimes \Phi_2, \exp(\xi) \otimes \exp(p)\rangle\rangle \\ &= (\mathcal{L}_1\Phi_1)(\xi)(\mathcal{L}_2\Phi_2)(p).\end{aligned}$$

If $\mathbf{1}$ is the function such that $\mathbf{1}(\omega) = 1$, $\forall \omega \in S'_{d.\mathbb{C}}$, then every element $V \in \mathcal{F}'_{\theta_2}(\mathbb{C}^r)$ can be identified with $V = \mathbf{1} \otimes V$ and moreover $(\mathcal{L}V)(\xi,p) = (\mathcal{L}_2 V)(p)$. The same reasoning can be applied to the convolution product, i.e., the convolution product $V * f$, $f \in \mathcal{F}'_{\theta_2}(\mathbb{C}^r)$ coincides with the usual convolution product with respect to the spatial variable.

3. Generalized stochastic calculus

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. Let $(B_t)_{t\geq 0}$ be a $\mathbb{R}^d$-valued $\mathcal{F}_t$-Brownian motion with $B_0 = 0$ P-a.s.

Later on, we need to define two types of integrals, deterministic and stochastic integrals of $\mathcal{F}'_\theta(\mathcal{N}')$-valued process. For the stochastic integrals we will use the theory of stochastic integration in Hilbert spaces developed in [DPZ92] and [Mét82]. Note that when Y_t is an $\mathcal{H}$-valued process, for a Hilbert space $\mathcal{H}$, then Y_t can be considered as an $L(\mathbb{R}^d, \mathbb{R}^d \otimes \mathcal{H})$-valued process, where $L(\cdot,\cdot)$ is the set of bounded linear operators and $\mathbb{R}^d \otimes \mathcal{H}$ is the tensor product of Hilbert spaces.

For any $(\omega, x) \in \mathcal{N}'$, $\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ we define $\tau_{(\omega,x)}\Phi \in \mathcal{F}'_\theta(\mathcal{N}')$ by

$$\langle\langle \tau_{(\omega,x)}\Phi, \varphi\rangle\rangle = \langle\langle \Phi, \tau_{-(\omega,x)}\varphi\rangle\rangle, \quad \varphi \in \mathcal{F}_\theta(\mathcal{N}').$$

For short we will denote $\tau_{(0,x)}$ by τ_x for any $x \in \mathbb{R}^r$.

Proposition 3.1 *Let $\Phi \in \mathcal{F}'_\theta(\mathbb{C}^r)$ and $g : \mathbb{R}^r \longrightarrow G_\theta(\mathbb{C}^r)$, $g(x) := (\mathcal{T} \circ \mathcal{L})(\tau_x\Phi)$. Then there exists $m > 0$ such that $g : \mathbb{R}^r \longrightarrow G_{\theta,m}(\mathbb{C}^r)$ is twice continuously differentiable and*

$$g'(x)(h) = -\sum_{i=1}^{r} (\mathcal{T} \circ \mathcal{L})(\partial_i \tau_x \Phi) h_i \tag{11}$$

$$g''(x)(h \otimes h) = \sum_{i,j=1}^{r} (\mathcal{T} \circ \mathcal{L})(\partial^2_{ij} \tau_x \Phi) h_i h_j. \tag{12}$$

Proof. If $\Phi \in \mathcal{F}'_\theta(\mathbb{C}^r)$, then $\tau_x\Phi \in \mathcal{F}'_\theta(\mathbb{C}^r)$. Since $\mathcal{T} \circ \mathcal{L}$ is a topological isomorphism between $\mathcal{F}'_\theta(\mathbb{C}^r)$ and $G_\theta(\mathbb{C}^r)$, there exist $m > 0$ such that $g(x)$ belongs to the Hilbert space $G_{\theta,m}(\mathbb{C}^r)$ for any $x \in \mathbb{R}^r$. In order to show the differentiability of g it is enough to obtain the following estimate

$$|g(x+h) - g(x) + \mathcal{T} \circ \mathcal{L} \sum_{i=1}^{r} (\partial_i \tau_x \Phi) h_i|_{G_{\theta,m}(\mathbb{C}^r)} \leq C(\theta, m, r) |g|_{G_{\theta,m'}(\mathbb{C}^r)} |h|^2,$$

for a certain $m' > 0$. Since $\mathcal{T}$ is an isomorphism between $\mathcal{G}_{\theta^*}(\mathbb{C}^r)$ and $G_\theta(\mathbb{C}^r)$ it is sufficient to show that

$$|\mathcal{L}(\tau_{x+h}\Phi) - \mathcal{L}(\tau_x\Phi) + \mathcal{L}\sum_{i=1}^{r}(\partial_i\tau_x\Phi)h_i|_{\mathcal{G}_{\theta^*,m}(\mathbb{C}^r)} \leq C(\theta,m,r)|\mathcal{L}(\tau_x\Phi)|_{\mathcal{G}_{\theta^*,m'}(\mathbb{C}^r)}|h|^2.$$

On one hand for $p \in \mathbb{C}^r$ we have $(\mathcal{L}\tau_x\Phi)(p) = (\mathcal{L}\Phi)(p)e^{(p,x)}$. On the other hand, the mapping $e^{(p,\cdot)} : \mathbb{R}^r \longrightarrow \mathbb{C}$, $x \mapsto e^{(p,x)}$ is differentiable and we have the following simple estimate

$$\left|\left(\mathcal{L}(\tau_{x+h}\Phi) - \mathcal{L}(\tau_x\Phi) + \mathcal{L}\sum_{i=1}^{r}(\partial_i\tau_x\Phi)h_i\right)(p)\right| \leq |\mathcal{L}(\tau_x\Phi)(p)||p|^2|h|^2e^{|p||h|}.$$

Since the Young function θ is such that $\limsup_{u\to\infty}\frac{\theta(u)}{u^2} < \infty$ which is equivalent to (see for example [KR61])

$$\liminf_{u\to\infty}\frac{\theta^*(u)}{u^2} > 0.$$

Hence there exist $u_0 > 0$ and $\varepsilon > 0$ such that $\theta^*(u) \geq \varepsilon u^2$ for $u \geq u_0$. We can choose $\alpha > 0$ and m' with $m > m' > 0$ such that $|h||p| \leq |h|^2/\alpha + \alpha|p|^2$ and

$$|p|^2e^{|h||p|}e^{-\theta^*(m|p|)} \leq C(\theta,m,r)e^{|h|^2/\alpha}e^{-\theta^*(m'|p|)}.$$

Therefore, we arrive at the following estimate

$$|\mathcal{L}(\tau_{x+h}\Phi) - \mathcal{L}(\tau_x\Phi) + \mathcal{L}\sum_{i=1}^{r}(\partial_i\tau_x\Phi)h_i|_{\mathcal{G}_{\theta^*,m}(\mathbb{C}^r)} \leq C(\theta,m,r)e^{|h|^2/\alpha}|\mathcal{L}(\tau_x\Phi)|_{\mathcal{G}_{\theta^*,m'}(\mathbb{C}^r)}|h|^2$$

which proves (11). The second derivative (12) is proved similarly.

Proposition 3.2 *Let $(X_s)_{0\leq s\leq T}$ be a given $\mathcal{F}'_\theta(\mathcal{N}')$-valued, $\mathcal{F}_t$-adapted continuous stochastic process.*

1. *We define the generalized stochastic process*

$$Y_t = \int_0^t X_s ds \in \mathcal{F}'_\theta(\mathcal{N}')$$

by

$$\mathcal{L}\left(\int_0^t X_s ds\right)(\xi,p) := \int_0^t (\mathcal{L}X_s)(\xi,p)ds,$$

where the right hand side integral is a Bochner integral. Moreover, Y_t is differentiable in $\mathcal{F}'_\theta(\mathcal{N}')$ and we have $\frac{\partial}{\partial t}Y_t = X_t$.

2. *Assume that there exists $m \in (\mathbb{R}_+^*)^2$ and $n \in \mathbb{N}_0$ such that $\mathcal{T}\circ\mathcal{L}X_s \in G_{\theta^*,m}(\mathcal{N}_{-n})$ and*

$$P\left(\int_0^T |\mathcal{T}\circ\mathcal{L}X_s|^2_{G_{\theta,m}(\mathcal{N}_{-n})}ds < \infty\right) = 1, \qquad (13)$$

then the stochastic integral

$$\int_0^t X_s dB_s \in \mathcal{F}'_\theta(\mathcal{N}')$$

is defined by

$$\mathcal{T}\left(\mathcal{L}\left(\int_0^t X_s dB_s\right)(\xi,p)\right) := \int_0^t \mathcal{T}\left((\mathcal{L}X_s)(\xi,p)\right) dB_s. \tag{14}$$

Proof. The first part of the proposition is proved in [OS02, Proposition 11]. Concerning the second part we notice that the mapping $s \mapsto \mathcal{T}\mathcal{L}X_s \in G_{\theta^*,m}(\mathcal{N}_{-n})$ is continuous. The condition (13) is sufficient for the existence of the stochastic integral (14), see for example [DPZ92] or [Mét82]. The proposition is proved.

It follows that the processes

$$\left(\int_0^t X_s dB_s\right)_{t\geq 0}, \qquad \left(\int_0^t \partial_i X_s dB_s\right)_{t\geq 0}$$

are $\mathcal{F}'_\theta(\mathcal{N}')$-valued continuous local martingales. If $Z_t = (Z_t^1, \dots, Z_t^d)$ is a $\mathbb{R}^d$-valued continuous $\mathcal{F}_t$-semi-martingale it follows from the general theory that the above processes with Z_t replacing B_t are $\mathcal{F}'_\theta(\mathcal{N}')$-valued continuous $\mathcal{F}_t$-semi-martingales.

Remark 3.3 Let $(X_s)_{0\leq s\leq T}$ be a given $\mathcal{F}'_\theta(\mathcal{N}')$-valued stochastic process such that the stochastic integral with respect to Z_t exists. Then for any $\varphi \in \mathcal{F}_\theta(\mathcal{N}')$ almost surely we have

$$\left\langle\!\!\left\langle \int_0^t X_s dZ_s, \varphi \right\rangle\!\!\right\rangle = \int_0^t \langle\!\langle X_s, \varphi\rangle\!\rangle dZ_s.$$

In fact, choosing φ from $\{\exp((\xi,p)),\, (\xi,p) \in \mathcal{N}\}$ which forms a total set in $\mathcal{F}_\theta(\mathcal{N}')$, we have

$$\begin{aligned}
\int_0^t \mathcal{T}\langle\!\langle X_s, \exp((\xi,p))\rangle\!\rangle dZ_s &= \int_0^t \mathcal{T}(\mathcal{L}X_s)(\xi,p) dZ_s \\
&= \mathcal{T}\left(\left(\mathcal{L}\int_0^t X_s dZ_s\right)(\xi,p)\right) \\
&= \mathcal{T}\left\langle\!\!\left\langle \int_0^t X_s dZ_s, \exp((\xi,p))\right\rangle\!\!\right\rangle.
\end{aligned}$$

On the other hand,

$$\int_0^t \mathcal{T}\langle\!\langle X_s, \exp((\xi,p))\rangle\!\rangle dZ_s = \mathcal{T}\int_0^t \langle\!\langle X_s, \exp((\xi,p))\rangle\!\rangle dZ_s,$$

since the integral $\int_0^t \langle\!\langle X_s, \varphi\rangle\!\rangle dZ_s$ is given by approximation and $\mathcal{T}$ is a topological isomorphism between $\mathcal{G}_{\theta^*}(\mathcal{N})$ and $G_\theta(\mathcal{N}')$.

The following theorem is an adaptation of Corollary 2.2 of [RT03] in our framework.

Theorem 3.4 *Let $(X_t)_{t\geq 0}$ be a given $\mathbb{R}^r$-valued continuous $\mathcal{F}_t$-semi-martingale and $\Phi \in \mathcal{F}'_\theta(\mathbb{C}^r)$. Then $\tau_{X_t}\Phi$ is an $\mathcal{F}'_\theta(\mathbb{C}^r)$-valued continuous $\mathcal{F}_t$-semi-martingale which has the following decomposition*

$$\tau_{X_t}\Phi = \tau_{X_0}\Phi - \sum_{i=1}^r \int_0^t \partial_i(\tau_{X_s}\Phi) dX_s^i + \frac{1}{2}\sum_{i,j=1}^r \int_0^t \partial_{ij}^2(\tau_{X_s}\Phi) d\langle X^i, X^j\rangle_s$$

where $\langle X^i, X^j\rangle_t$ is the quadratic variation process between X_t^i and X_t^j, $1 \leq i,j \leq r$.

Proof. For $\Phi \in \mathcal{F}'_\theta(\mathbb{C}^r)$ we have $\tau_{X_t}\Phi \in \mathcal{F}'_\theta(\mathbb{C}^r)$. By Proposition 3.1 there exists $m > 0$ such that the mapping

$$g : \mathbb{R}^r \longrightarrow G_{\theta,m}(\mathbb{C}^r),\ x \mapsto g(x) := (\mathcal{T} \circ \mathcal{L})(\tau_x \Phi)$$

is twice continuously Fréchet differentiable and, therefore applying the Itô formula yields

$$g(X_t) = g(X_0) + \sum_{i=1}^r \int_0^t \partial_i g(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^r \int_0^t \partial_{ij}^2 g(X_s) d\langle X^i, X^j \rangle_s.$$

Then the result of the theorem follows from (11), (12) and applying $(\mathcal{T} \circ \mathcal{L})^{-1}$. The theorem is proved.

Proposition 3.5 *Let $\Phi, \Psi_t \in \mathcal{F}'_\theta(\mathbb{C}^r)$, $t \geq 0$ be given, $(X_t)_{t\geq 0}$ a $\mathbb{R}^r$-valued continuous $\mathcal{F}_t$-semi-martingale and define $Z_t := \exp^*(\int_0^t \Psi_s ds)$. Then $\Xi_t := \tau_{X_t}(Z_t * \Phi)$ is a continuous $\mathcal{F}_t$-semi-martingale which has the following decomposition*

$$\Xi_t = \Xi_0 + \int_0^t \Psi_s * \Xi_s ds - \sum_{i=1}^r \int_0^t \partial_i \Xi_s dX_s^i + \frac{1}{2} \sum_{i,j=1}^r \int_0^t \partial_{ij}^2 \Xi_s d\langle X^i, X^j \rangle_s.$$

Proof. Let $Y_t \in \mathcal{F}'_\theta(\mathbb{C}^r)$, $t \in [0,T]$ be a given process of class C^1 in $\mathcal{F}'_\theta(\mathbb{C}^r)$ and consider

$$f : [0,T] \times \mathbb{R}^r \longrightarrow G_\theta(\mathbb{C}^r),\ f(t,x) := (\mathcal{T} \circ \mathcal{L})(\tau_x Y_t).$$

In the same way as in Proposition 3.1 there exist $m > 0$ such that $f : [0,T] \times \mathbb{R}^r \longrightarrow G_{\theta,m}(\mathbb{C}^r)$ is of class C^1 in t and of class C^2 in x.

Applying the Itô formula gives

$$\begin{aligned} f(t, X_t) &= f(0, X_0) + \int_0^t \partial_t f(s, X_s) ds - \sum_{i=1}^r \int_0^t \partial_i f(s, X_s) dX_s^i \\ &\quad + \frac{1}{2} \sum_{i,j=1}^r \int_0^t \partial_{ij}^2 f(s, X_s) d\langle X^i, X^j \rangle_s. \end{aligned}$$

For the case $Y_t = Z_t * \Phi$ we only need to show that $\partial_t f(s, X_s) = (\mathcal{T} \circ \mathcal{L})(\Xi_s * \Psi_s)$. This follows from the fact that $\partial_t Z_t = \Psi_t * Z_t$ by using the Laplace transform. Finally, the result follows as in the previous theorem.

4. GENERALIZED FEYNMAN-KAC FORMULA

We consider the following Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t} U_t = \frac{1}{2} \Delta U_t + V_t * U_t \\ u_0 = f. \end{cases} \tag{15}$$

The different terms in (15) are as follows: $\Delta = \sum_{i=1}^r \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the generalized sense on $\mathbb{R}^r$, the initial conditions $f \in \mathcal{F}'_\theta(\mathbb{R}^r)$ and V_t are $\mathcal{F}'_\theta(\mathbb{R}^r)$-valued continuous generalized functions.

The aim of this section is to give a probabilistic representation for the solution of the Cauchy problem (15). Before we give two general lemmas. In the following β denotes the Young function $(e^{\theta^*})^*$.

Lemma 4.1 *Let $\Phi \in \mathcal{F}'_\beta(\mathbb{C}^r)$, $x \in \mathbb{R}^r$ be given such that $\mathcal{T} \circ \mathcal{L}(\tau_x \Phi) \in G_{\beta,m}(\mathbb{C}^r)$. Then there exist $C(\alpha, \theta, m), m' > 0$ such that*

$$|\mathcal{T} \circ \mathcal{L}(\partial_i(\tau_x \Phi))|_{G_{\beta,m}(\mathbb{C}^r)} \leq C |\mathcal{T} \circ \mathcal{L}(\Phi)|_{G_{\beta,m'}(\mathbb{C}^r)} e^{|x|^2/\alpha},$$

for any $\alpha > 0$.

Proof. Since $\mathcal{T}$ is an isomorphism between $\mathcal{G}_{\beta^*}(\mathbb{C}^r)$ and $G_\beta(\mathbb{C}^r)$ it is sufficient to show that

$$|\mathcal{L}(\partial_i(\tau_x \Phi))|_{\mathcal{G}_{\beta^*,m}(\mathbb{C}^r)} \leq C |\mathcal{L}(\Phi)|_{\mathcal{G}_{\beta^*,m'}(\mathbb{C}^r)} e^{|x|^2/\alpha}.$$

For any $p \in \mathbb{C}^r$ we have

$$|\mathcal{L}(\partial_i(\tau_x \Phi))(p)| \leq |\mathcal{L}(\Phi)(p)||p| e^{|p||x|}.$$

Since the Young function θ verifies $\limsup_{u \to \infty} \frac{\theta(u)}{u^2} < \infty$, then there exist $u_0 > 0$ and $\varepsilon > 0$ such that $\theta^*(u) \geq \varepsilon u^2$ for $u \geq u_0$. Hence $\beta^* = e^{\theta^*}$ satisfies $\beta^*(u) \geq e^{\varepsilon u^2}$, for $u \geq u_0$.

Now we can easily see that there exist $C > 0$ and m' with $m > m' > 0$ such that

$$|p| e^{|x||p|} e^{-\beta^*(m|p|)} \leq C e^{-\beta^*(m'|p|)} e^{|x|^2/\alpha}.$$

This implies the result.

Let us recall that if $\Phi \in \mathcal{F}'_\theta(\mathbb{C}^r)$, we have $\exp^*(\Phi) \in \mathcal{F}'_\beta(\mathbb{C}^r)$ with $\beta = (e^{\theta^*})^*$, see for example Lemma 6 in [OS02].

Lemma 4.2 *Let $f, V_t \in \mathcal{F}'_\theta(\mathbb{C}^r)$ be such that the process Φ_t, $t \in [0,T]$ defined by*

$$\Phi_t := \left(f * \exp^* \left(\int_0^t V_s ds \right) \right)$$

has the property $(\mathcal{T} \circ \mathcal{L})(\Phi_t) \in G_{\beta,m}(\mathbb{C}^r)$. Then the following stochastic integral

$$\left\{ \int_0^t \mathcal{T} \circ \mathcal{L}(\partial_i(\tau_{B_s} \Phi_s)) dB_s^i, \quad t \leq T \right\},$$

is a $L^2(P)$-bounded martingale.

Proof. In order to prove the required martingale property it is sufficient to show (cf. [KS91]) that there exists $m > 0$ such that

$$\mathbf{E} \left(\int_0^T |\mathcal{T} \circ \mathcal{L}(\partial_i(\tau_{B_t} \Phi_t))|^2_{G_{\beta,m}(\mathbb{C}^r)} dt \right) < \infty,$$

or equivalently

$$\mathbf{E} \left(\int_0^T |\mathcal{L}(\partial_i(\tau_{B_t} \Phi_t))|^2_{\mathcal{G}_{\beta^*,m}(\mathbb{C}^r)} dt \right) = \int_{\mathbb{R}^r} \int_0^T |\mathcal{L}(\partial_i(\tau_x \Phi_t))|^2_{\mathcal{G}_{\beta^*,m}(\mathbb{C}^r)} p_t(x) dt dx < \infty,$$

where $p_t(x)$ is the Gaussian density in $\mathbb{R}^r$, i.e., $p_t(x) = \frac{1}{(2\pi t)^{r/2}} e^{-|x|^2/2t}$. It follows from Lemma 4.1 that for $\alpha > 2T$ we have

$$\begin{aligned} & \int_{\mathbb{R}^r} \int_0^T |\mathcal{L}(\partial_i(\tau_x \Phi_t))|^2_{\mathcal{G}_{\beta^*,m}(\mathbb{C}^r)} p_t(x) dt dx \\ \leq \ & C \int_0^T |\mathcal{L}(\Phi_t)|_{\mathcal{G}_{\beta^*,m'}(\mathbb{C}^r)} \int_{\mathbb{R}^r} e^{|x|^2/\alpha} p_t(x) dx dt \\ \leq \ & \frac{C}{2^{r/2}} \int_0^T \frac{1}{(t(\frac{1}{2t} - \frac{1}{\alpha}))^{r/2}} dt < \infty, \end{aligned}$$

where C is a constant which change from line to line.

Theorem 4.3 *The solution of the Cauchy problem (15) is given by*

$$U_t = \mathbf{E}^x \left(\tau_{B_t} \left(f * \exp^* \left(\int_0^t V_s ds \right) \right) \right),$$

where $(B_t)_{t \geq 0} = (B_t^1, \ldots, B_t^r)$ is a $\mathbb{R}^r$-valued Brownian motion with probability law P^x when starting at $B_0 = x \in \mathbb{R}^r$. $\mathbf{E}^x$ denotes the expectation with respect to P^x.

Proof. Let us denote by Θ_t, $t \geq 0$ the process

$$\Theta_t := \tau_{B_t} \left(f * \exp^* \left(\int_0^t V_s ds \right) \right).$$

At first we notice that $\mathcal{L}\left(f * \exp^* \left(\int_0^t V_s ds \right) \right)$ can be extended to an entire function on $\mathbb{C}^r$ which satisfies the estimate

$$\left| \mathcal{L} \left(f * \exp^* \left(\int_0^t V_s ds \right) \right)(p) \right| \leq C e^{\beta^*(m|p|)}, \qquad \forall p \in \mathbb{C}^r$$

which implies that $f * \exp^* \left(\int_0^t V_s ds \right) \in \mathcal{G}_{\beta^*}(\mathbb{C}^r)$, cf. Remark (2) in [GHOR00]. Since B is a $\mathbb{R}^r$-valued Brownian motion, then $\langle B^i, B^j \rangle = 0$ for $i \neq j$. It follows from Proposition 3.5 that the process Θ_t has the following decomposition

$$\begin{aligned} \Theta_t &= f - \sum_{i=1}^r \int_0^t \partial_i \Theta_s dB_s^i + \frac{1}{2} \sum_{i=1}^r \int_0^t \partial_{ii}^2 \Theta_s ds \\ &\quad + \int_0^t V_s * \Theta_s ds. \end{aligned}$$

Taking expectation in the the above equality, and the fact that the stochastic integral is a martingale, cf. Lemma 4.2, yields

$$U_t = f + \frac{1}{2} \sum_{i=1}^r \int_0^t \partial_i^2 U_s ds + \int_0^t V_s * U_s ds,$$

which satisfies the Cauchy problem (15). The theorem is proved.

Now we consider the heat equation with stochastic potential

$$\begin{cases} \frac{\partial}{\partial t}X_t = \frac{1}{2}\Delta X_t + V_t * X_t \\ \\ X_0 = f, \end{cases} \tag{16}$$

where $X_t, f \in \mathcal{F}'_\theta(\mathcal{M}')$, $t \geq 0$ is the time parameter and $\Delta = \sum_{i=1}^r \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in the generalized sense on $\mathbb{R}^r$, $\omega = (\omega_1, \ldots, \omega_d)$ is the stochastic vector variable in the tempered Schwartz distribution space $S'(\mathbb{R}, \mathbb{R}^d)$, $d \in \mathbb{N}$, and $*$ is the convolution product between generalized functions on $\mathcal{F}'_\theta(S'(\mathbb{R}, \mathbb{R}^d) \otimes \mathbb{R}^r)$.

In order to solve the Cauchy problem (16) we only need to adapt the result of Proposition 3.1. To this end, we consider

$$g : \mathbb{R}^r \longrightarrow G_\theta(\mathcal{N}'),\ g(x) := (\mathcal{T} \circ \mathcal{L})(\tau_{(0,x)}\Phi), \qquad \Phi \in \mathcal{F}'_\theta(\mathcal{N}').$$

Then there exists $m \in (\mathbb{R}_+^*)^2$ and $n \in \mathbb{N}_0$ such that $g : \mathbb{R}^r \longrightarrow G_{\theta,m}(\mathcal{N}_{-n})$ is twice continuously differentiable and

$$\begin{aligned} g'(x)(h) &= -\sum_{i=1}^r (\mathcal{T} \circ \mathcal{L})(\partial_i \tau_{(0,x)}\Phi)h_i \\ g''(x)(h \otimes h) &= \sum_{i,j=1}^r (\mathcal{T} \circ \mathcal{L})(\partial_{ij}^2 \tau_{(0,x)}\Phi)h_i h_j. \end{aligned}$$

The next theorem can be shown in the same way as Theorem 4.3.

Theorem 4.4 *There exists a unique generalized stochastic process X_t which solves the Cauchy problem (16), namely*

$$X_t = \tilde{\mathbf{E}}^x \left(\tau_{(0,\tilde{B}_t)} \left(f * \exp^* \left(\int_0^t V_s ds \right) \right) \right), \tag{17}$$

where $\tilde{B}_t$, $t \geq 0$ is a $\mathbb{R}^r$-valued Brownian motion starting at $\tilde{B}_0 = x \in \mathbb{R}^r$.

Next we will study a modification of the Cauchy problem (16). More precisely, we consider

$$\begin{cases} \frac{\partial}{\partial t}X_t = \frac{1}{2}\Delta X_t + H_t * \nabla X_t \\ \\ X_0 = f, \qquad t \in [0, T]. \end{cases} \tag{18}$$

where $f \in \mathcal{F}'_\theta(\mathcal{M}')$, $H_t \in \mathcal{F}'_{\theta_2}(S'_d)$ and the convolution product $*$ concerns only with the distributional variable $\omega \in S'_d$. Clearly, the Laplacian Δ and the gradient ∇ in the generalized sense is with respect to the spatial variable $x \in \mathbb{R}^r$. Applying the Laplace transform $\mathcal{L}_1$ (cf. Remark 2.7) to (18) we obtain

$$\begin{cases} \frac{\partial}{\partial t}(\mathcal{L}_1 X_t)(\xi) = \frac{1}{2}\Delta(\mathcal{L}_1 X_t)(\xi) + (\mathcal{L}_1 H_t)(\xi)\nabla(\mathcal{L}_1 X_t)(\xi) \\ \\ (\mathcal{L}_1 X_0)(\xi) = (\mathcal{L}_1 f)(\xi), \qquad \xi \in S_d. \end{cases} \tag{19}$$

Notice that, $\mathcal{L}_1 f \in \mathcal{F}'_{\theta_1}(\mathbb{R}^r)$, in other words, generalized functions in the spatial variable. In order to solve (19) we consider a $\mathbb{R}^r$-valued continuous semi-martingale, namely

$$dY_t = dB_t + (\mathcal{L}_1 H_t)(\xi)dt,$$

where B_t is a Brownian motion with probability law P^x when starting at $B_0 = x \in \mathbb{R}^r$. The solution of (19) is given by the generalized Feynman-Kac formula as (cf. Theorem 4.3)

$$(\mathcal{L}_1 X_t)(\xi) = \mathbf{E}^{\,x}\left(\tau_{(0,Y_t)}(\mathcal{L}_1 f)(\xi)\right).$$

Using the Girsanov transformation the semi-martingale Y_t, $t \geq 0$ is a Brownian motion with respect to the Probability measure $\tilde{P}^x$ such that

$$\frac{d\tilde{P}^x}{dP^x} = \exp\left(-\int_0^t (\mathcal{L}_1 H_s)(\xi)dB_s - \frac{1}{2}\int_0^t (\mathcal{L}_1 H_s)^2(\xi)ds\right).$$

Hence, the solution $(\mathcal{L}_1 X_t)$ becomes

$$(\mathcal{L}_1 X_t)(\xi) = \mathbf{E}^{\,\tilde{P}^x}\left(\left(\tau_{(0,Y_t)}(\mathcal{L}_1 f)(\xi)\right)\exp\left(\int_0^t (\mathcal{L}_1 H_s)(\xi)dY_s - \frac{1}{2}\int_0^t (\mathcal{L}_1 H_s)^2(\xi)ds\right)\right).$$

On one hand, we notice that $\mathcal{L}_1 X_t$ may be extended to an entire function on the complexification of S_d and therefore $\mathcal{L}X_t$ is an entire function on $\mathcal{N}$. Moreover, using the estimate

$$\begin{aligned}
&\left|\exp\left(\int_0^t (\mathcal{L}_1 H_s)(\xi)dY_s - \frac{1}{2}\int_0^t (\mathcal{L}_1 H_s)^2(\xi)ds\right)\right| \\
\leq\ & \exp\left(\int_0^t \mathrm{Re}(\mathcal{L}_1 H_s)(\xi)dY_s - \frac{1}{2}\int_0^t (\mathrm{Re}(\mathcal{L}_1 H_s)(\xi))^2 ds\right) \\
&\times \exp\left(\frac{1}{2}\int_0^t (\mathrm{Im}(\mathcal{L}_1 H_s)(\xi))^2 ds\right)
\end{aligned}$$

and the fact that $\mathcal{L}f \in \mathcal{G}_{\theta^*}(\mathcal{N})$, $\mathcal{L}_1 H_t \in \mathcal{G}_{\theta_2^*}(\mathbb{C}^r)$ there exist $m \in (\mathbb{R}_+^*)^2, n \in \mathbb{N}_0$ such that

$$\begin{aligned}
&|(\mathcal{L}X_t)(\xi,p)| \\
\leq\ & |(\mathcal{L}f)(\xi,p)|\exp\left(\frac{1}{2}\int_0^t (\mathrm{Im}(\mathcal{L}_1 H_s)(\xi))^2 ds\right) \\
&\times \mathbf{E}^{\,\tilde{P}^x}\left(|e^{(p,Y_t)}|\exp\left(\int_0^t \mathrm{Re}(\mathcal{L}_1 H_s)(\xi)dY_s - \frac{1}{2}\int_0^t (\mathrm{Re}(\mathcal{L}_1 H_s)(\xi))^2 ds\right)\right) \\
\leq\ & |(\mathcal{L}f)(\xi,p)|\exp\left(\frac{1}{2}\int_0^t |(\mathcal{L}_1 H_s)(\xi)|^2 ds\right)\left(\mathbf{E}^{\,\tilde{P}^x} e^{2\mathrm{Re}(p,Y_t)}\right)^{1/2} \\
\leq\ & |(\mathcal{L}f)(\xi,p)|\exp\left(\frac{1}{2}\int_0^t |(\mathcal{L}_1 H_s)(\xi)|^2 ds\right)e^{|p|^2 T} \\
\leq\ & Ce^{\beta^*(m|(\xi,p)|_n)}.
\end{aligned}$$

This implies that $\mathcal{L}X_t \in \mathcal{G}_{\beta^*}(\mathcal{N})$, cf. Remark (2) in [GHOR00].

Finally, the solution of (18) is given applying the inverse Laplace transform. We have proved the following theorem.

Theorem 4.5 *There exists a unique generalized stochastic process X_t which solves the Cauchy problem (18), namely*

$$X_t = \mathbf{E}^{\,\tilde{P}^x}\left(\left(\tau_{(0,Y_s)}f\right) * \exp^*\left(\int_0^t H_s dY_s - \frac{1}{2}\int_0^t H_s^{*2} ds\right)\right),$$

where Y_t, $t \geq 0$ is a $\mathbb{R}^r$-valued Brownian motion starting at $Y_0 = x \in \mathbb{R}^r$.

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