Preliminary version

Characterization of canonical marked Gibbs measures

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1 Introduction
2 Configuration spaces

In this section we describe the configuration spaces which is the framework to be used in the rest of the paper. In Subsection 2.1, we introduce the space of finite configurations $\hat{\Gamma}_0$ as a sum of manifolds. As a result we obtain a differentiable structure on $\hat{\Gamma}_0$. In Subsection 2.2, we define the space of configurations $\hat{\Gamma}$ and its measurable structure.

Let $X$ be a connected, oriented Riemannian $C^\infty$-manifold with volume element $m$. This is the physical position space of the particles. For simplicity we assume that $X$ is geodesically complete; in the most interesting applications, is non-compact. We denote by $\mathcal{O}(X)$ the family of all open subsets of $X$, and by $\mathcal{B}(X)$ the corresponding $\sigma$-algebra on $X$. $\mathcal{O}_c(X)$ and $\mathcal{B}_c(X)$ denote the system of all sets in $\mathcal{O}(\hat{X})$, $\mathcal{B}(\hat{X})$, respectively which are bounded (and hence have compact closure). There exists an order generating sequence $(\Lambda_n)_{n \in \mathbb{N}}$ in $\mathcal{O}_c(X)$, i.e., $\Lambda_n \subset \Lambda_{n+1}$, $\bigcup_{n \in \mathbb{N}} \Lambda_n = X$ and for every $\Lambda \in \mathcal{B}_c(X)$ there exists an $n \in \mathbb{N}$ such that $\Lambda \subset \Lambda_n$. In addition to the position space $X$, we consider another manifold $S$ with another volume element $n$. By $\mathcal{B}(S)$ we denote the corresponding Borel $\sigma$-algebra. The elements of this space we call marks; they can describe for instance: internal particle degrees of freedom.

We denote the product space $X \times S$ by $\hat{X}$. Analogously, we define $\mathcal{O}_c(\hat{X})$, $\mathcal{B}_c(\hat{X})$. We also need to introduce the family of “local sets” on $\hat{X}$ which are not as for $X$ the bounded sets, but all sets from

$$L(\hat{X}) := \{ \hat{\Lambda} \subset \hat{X} | \exists \Lambda \in \mathcal{B}_c(X) \ \hat{\Lambda} \subset \Lambda \times S \}.$$ 

Note that the treatment of the position and marks is assymetric. For simplicity of notation and when there is no danger of confusion we will denote elements of the form $\Lambda \times S$, $\Lambda \in \mathcal{B}_c(X)$, for short by $\Lambda$. Define $\mathcal{O}_c(\hat{X}) := \{ \Lambda \times S | \Lambda \in \mathcal{O}_c(X) \}$ and $\mathcal{B}_c(\hat{X}) := \{ \Lambda \times S | \Lambda \in \mathcal{B}_c(X) \}$.

2.1 Finite marked configuration space

Let $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $Y \in \mathcal{B}(\hat{X})$ be given. Define the space $\Gamma_Y^{(n)}$ of $n$-point marked configurations in $Y$ by

$$\Gamma_Y^{(n)} := \{ \hat{\eta} \subset Y | |\hat{\eta}| = n \}, \ \Gamma_Y^{(0)} := \{ \emptyset \},$$

(2.1)
where $|\hat{\eta}|$ denotes the cardinality of the set $\hat{\eta}$. For any $\Lambda \in \mathcal{L}(\hat{X})$ with $\Lambda \subset Y$, define the mapping $N_{\Lambda}$ by

$$N_{\Lambda} : \Gamma_{\hat{Y}}^{(n)} \to \mathbb{N}_0, \; \hat{\eta} \mapsto |\hat{\eta} \cap \Lambda|. \tag{2.2}$$

This will play an essential role on the construction. For short we denote by $\hat{\eta}_{\Lambda} := \hat{\eta} \cap \Lambda$. The space of all $n$-point configurations $\Gamma_{\hat{X}}^{(n)}$ possesses the structure of a $C^\infty$-manifold of dimension $n \cdot \dim(X) \cdot \dim(S)$. More precisely, let $\hat{Y}^n := \{(\hat{x}_1, \ldots, \hat{x}_n) \in Y^n | \hat{x}_i \neq \hat{x}_j \text{ if } i \neq j\}$, $\hat{x}_i = (x_i, s_i) \in X \times S$ be given. Then we may define the natural map $\text{sym}^n_{\hat{Y}} : \hat{Y}^n \to \Gamma_{\hat{X}}^{(n)}$ given by

$$\hat{Y}^n \ni (\hat{x}_1, \ldots, \hat{x}_n) \mapsto \text{sym}^n_{\hat{Y}}((\hat{x}_1, \ldots, \hat{x}_n)) := \{\hat{x}_1, \ldots, \hat{x}_n\}. \tag{2.3}$$

Using this mapping we may identify the space of $n$-point configuration $\Gamma_{\hat{X}}^{(n)}$ with the symmetrization of $\hat{Y}^n$, i.e., $\hat{Y}^n/S_n$, where $S_n$ is the symmetric group of permutations of $\{1, \ldots, n\}$. Then $\Gamma_{\hat{X}}^{(n)}$ inherits the structure of $(\hat{X} \times S)^n/S_n$ as a $C^\infty$-manifold of dimension $n \cdot \dim(X) \cdot \dim(S)$. The family of all open sets on $\Gamma_{\hat{X}}^{(n)}$ we denote by $\mathcal{O}(\Gamma_{\hat{X}}^{(n)})$ and the corresponding Borel $\sigma$-algebra by $\mathcal{B}(\Gamma_{\hat{X}}^{(n)})$. This $\sigma$-algebra coincides with the $\sigma$-algebra generated by the mappings $N_{\Lambda}$, i.e.,

$$\mathcal{B}(\Gamma_{\hat{X}}^{(n)}) = \sigma(N_{\Lambda} | \Lambda \in \mathcal{L}(\hat{X})). \tag{2.4}$$

Notice that a basis for the topology on $\Gamma_{\hat{X}}^{(n)}$ is given by the following family of sets

$$\{\hat{U}_1 \hat{\times} \ldots \hat{\times} \hat{U}_n, \hat{U}_i \in \mathcal{O}_c(\hat{X}), \hat{U}_i \cap \hat{U}_j = \emptyset, \; i \neq j, \; i = 1, \ldots, n\}, \tag{2.5}$$

where

$$\hat{U}_1 \hat{\times} \ldots \hat{\times} \hat{U}_n := \{\hat{\eta} \in \Gamma_{\hat{X}}^{(n)} | N_{\hat{U}_1}(\hat{\eta}) = 1, \ldots, N_{\hat{U}_n}(\hat{\eta}) = 1\}.$$
is a chart of \( \Gamma^{(n)}_X \). If \( Y \) is a submanifold of \( \hat{X} \) then \( \Gamma^{(n)}_{\hat{Y}} \) is a submanifold of \( \Gamma^{(n)}_X \). Finally, we define the space of finite configurations \( \hat{\Gamma}_0 := \Gamma_{0,\hat{X}} \) as

\[
\Gamma_{0,\hat{X}} := \bigsqcup_{n \in \mathbb{N}_0} \Gamma^{(n)}_X
\]

equipped with the topology \( \mathcal{O}(\Gamma_{0,\hat{X}}) \) of disjoint union.

### 2.2 Configuration space

The configuration space \( \hat{\Gamma} := \Gamma_{\hat{X}} \) over the space \( \hat{X} \) is defined as the set of all locally finite subsets (configurations) of \( \hat{X} \):

\[
\hat{\Gamma} := \{ \hat{\gamma} \subset \hat{X} \mid |\hat{\gamma} \cap K| < \infty \text{ for any compact } K \subset \hat{X} \}.
\]

(2.7)

The space \( \hat{\Gamma} \) is equipped with the vague topology \( \mathcal{O}(\hat{\Gamma}) \), i.e., the weakest topology such that all functions \( \hat{\Gamma} \to \mathbb{R}, \hat{\gamma} \mapsto \langle f, \hat{\gamma} \rangle := \sum_{\hat{x} \in \hat{\gamma}} f(\hat{x}) \) are continuous for any \( f \in C_{bs}(\hat{X}) \) (the set of all continuous functions on \( \hat{X} \) with bounded support). The topology \( \mathcal{O}(\hat{\Gamma}) \) is separable and metrizable, see e.g., [KMM78] and moreover \( \hat{\Gamma} \) is a Polish space. A sequence \( (\hat{\gamma}_n)_{n \in \mathbb{N}} \) converges in \( \mathcal{O}(\hat{\Gamma}) \) to \( \hat{\gamma} \) iff \( N_\Lambda(\hat{\gamma}_n) \to N_\Lambda(\hat{\gamma}) \) for all \( \Lambda \in \mathcal{L}(\hat{X}) \) with \( N_{\partial \Lambda}(\hat{\gamma}) = 0 \), where \( \partial \Lambda \) denotes the topological boundary of \( \Lambda \). The Borel \( \sigma \)-algebra corresponding to \( \mathcal{O}(\hat{\Gamma}) \) we denote by \( \mathcal{B}(\hat{\Gamma}) \). This \( \sigma \)-algebra coincides with the smallest \( \sigma \)-algebra for which all the mappings

\[
N_\Lambda : \hat{\Gamma} \to \mathbb{N}_0, \quad \hat{\gamma} \mapsto |\hat{\gamma} \cap \Lambda|, \quad \Lambda \in \mathcal{L}(\hat{X}),
\]

are measurable, i.e., \( \mathcal{B}(\hat{\Gamma}) = \sigma(N_\Lambda \mid \Lambda \in \mathcal{L}(\hat{X})) \). We can also give an alternative description of the configuration space \( \hat{\Gamma} \) as the projective limit of a certain family of measurable spaces. To this end let \( \Lambda \in \mathcal{L}(\hat{X}) \) be a given bounded open set. Define \( \Gamma_\Lambda \) by

\[
\Gamma_\Lambda := \{ \hat{\gamma} \in \hat{\Gamma} \mid \hat{\gamma}_{\hat{X} \setminus \Lambda} = \emptyset \}.
\]

Notice that we can write \( \Gamma_\Lambda \) in terms of \( \Gamma^{(n)}_\Lambda, n \in \mathbb{N}_0 \) as follows

\[
\Gamma_\Lambda = \bigsqcup_{n \in \mathbb{N}_0} \Gamma^{(n)}_\Lambda
\]
and equipped it with the $\sigma$-algebra of disjoint union $B(\Gamma_\Lambda)$. For any $\Lambda_1, \Lambda_2 \in \mathcal{O}_c(\hat{X})$ with $\Lambda_1 \subset \Lambda_2$ there is a natural mapping

$$p_{\Lambda_2, \Lambda_1} : \Gamma_{\Lambda_2} \to \Gamma_{\Lambda_1}, \hat{\gamma} \mapsto \hat{\gamma}_{\Lambda_1}. \quad (2.9)$$

This mapping is $B(\Gamma_{\Lambda_2})$-$B(\Gamma_{\Lambda_1})$-measurable. The space of configurations $\hat{\Gamma}$ is nothing else then the projective limit of the family $(\Gamma_\Lambda)_{\Lambda \in B(\hat{X})}$ with respect to the projections (2.9) and the measurable mappings

$$p_\Lambda : \hat{\Gamma} \to \Gamma_\Lambda, \hat{\gamma} \mapsto \hat{\gamma}_\Lambda. \quad (2.10)$$

Furthermore, for any $\Lambda \in \mathcal{L}(\hat{X})$ we define the following $\sigma$-algebra on $\hat{\Gamma}$

$$B_\Lambda(\hat{\Gamma}) := \sigma(\{N_{\Lambda'} | \Lambda' \in \mathcal{L}(\hat{X}) \text{ with } \Lambda' \subset \Lambda\}). \quad (2.11)$$

The $\sigma$-algebras $B_\Lambda(\hat{\Gamma})$ and $B(\Gamma_\Lambda)$ are $\sigma$-isomorphic, i.e., there exists a bijective mapping between them which preserves the operations of a $\sigma$-algebra. The filtration $(B_\Lambda(\hat{\Gamma}))_{\Lambda \in \mathcal{L}(\hat{X})}$ is one of the basic structures in the definition of Gibbs measures, see Section 3 below. For more details concerning configurations space analysis and its applications we refer to the works of Albeverio et al. [AKR98a], [AKR98b] and [Kun99] or [KK02a] and references therein.
3 Measures on configuration spaces

In this section we introduce the main objects of our considerations later on. Namely, the “free” measures on the configuration space $\hat{\Gamma}$ and $\hat{\Gamma}_0$, Gibbs measures as probability measures on $\hat{\Gamma}$ and as dual objects the correlation measures on $\hat{\Gamma}_0$ (the states of our physical system). This is introduced in Subsections 3.1, 3.2, and 3.4. In Subsection 3.3 we introduce an useful transformation, so-called $K$-transform which connects probability measures on the configuration space $\hat{\Gamma}$ and correlations measures on $\hat{\Gamma}_0$, cf. Theorem 3.16 below. This transform also allows us to control the properties of the Radon-Nikodym and the logarithemic derivative of the measures.

3.1 Marked Poisson and marked Lebesgue-Poisson measures

First of all we fix an intensity measure $\sigma$ on the manifold $\hat{X}$. Recall, that we denote by $m$ a volume element on $X$ and by $n$ a volume element on $S$. Let $\hat{\rho}$ be a non-negative function on $X \times S$ such that for all $\Lambda \in L(\hat{X})$

$$\int_{\Lambda} \hat{\rho}(x, s) n(ds)m(dx) < \infty. \quad (3.12)$$

We define $\sigma(dx, ds) := \hat{\rho}(x, s)n(ds)m(dx)$ and $\hat{m} := n \otimes m$. Our examples are interesting for the case $\sigma(\hat{X}) = \infty$ which we thus assume. For the part of characterization via Radon-Nikodym derivative we have to assume that $\hat{\rho} > 0 \hat{m}$-a.s. For the case of integration by parts we have to assume that $\hat{\rho} \in L^1(\hat{X}, \hat{m})$ and that there exists a vector field $\beta_\sigma$ locally integrable and continuous such that $\nabla \hat{\rho}(\hat{x}) = \hat{\rho}(\hat{x})\beta_\sigma(\hat{x})$. In particular this holds for $\sqrt{\hat{\rho}} \in H^{1,2}_{\text{loc}}(\hat{X}, \hat{m})$ (the Sobolev space of order $(1,2)$).

For any $Y \in \mathcal{B}(\hat{X})$, $n \in \mathbb{N}$, and for the product measure $\sigma^{\otimes n}$ the set $\hat{Y}^n$ differs only by a set of $\sigma^{\otimes n}$-measure zero from $Y^n$. The projection of this measure on $\Gamma_{0,Y}^{(n)}$ via $\text{sym}_Y^n$ we denote by $\sigma_n$, i.e.,

$$\sigma_n \mid \Gamma_{0,Y}^{(n)} := \sigma^{\otimes n} \circ (\text{sym}_Y^n)^{-1}.$$

On $\Gamma_{0,Y}^{(0)}$ the measure $\sigma_0$ is given by $\sigma_0(\emptyset) := 1$. The marked Lebesgue-Poisson measure $\lambda_{z,\sigma}$ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is defined by

$$\lambda_{z,\sigma} := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma_n. \quad (3.13)$$
As a result, \( \lambda_{z\sigma} \) is \( \sigma \)-finite measure. \( z > 0 \) is the so-called activity parameter.

Considered as a measure on \( \Gamma_{\Lambda}, \Lambda \in \mathcal{O}_c(\hat{X}) \), the measure \( \lambda_{z\sigma} \) is finite with \( \lambda_{z\sigma}(\Gamma_\Lambda) = e^{z\sigma(\Lambda)} \). Therefore, we can define a probability measure \( \pi_{z\sigma}^\Lambda \) on \( \Gamma_\Lambda \) putting

\[
\pi_{z\sigma}^\Lambda := e^{-z\sigma(\Lambda)} \lambda_{z\sigma}.
\]

(3.14)

Since the family \( \{ \pi_{z\sigma}^\Lambda | \Lambda \in \mathcal{O}_c(\hat{X}) \} \) is consistent, by a version of Kolmogorov’s theorem for projective limit spaces (cf. [Par67, Chap. V Theorem 3.2]) such a family determines uniquely a measure \( \pi_{z\sigma} \) on \( \mathcal{B}(\hat{\Gamma}) \). The measure \( \pi_{z\sigma} \) is called marked Poisson measure.

**Remark 3.1** We will explain the label “marked” later in this section, cf. Remark 3.18 and Remark 3.20 below.

It is possible to compute the Laplace transform of \( \pi_{z\sigma} \). In fact, for a given \( f \in C_0(\hat{X}) \) (the set of continuous functions on \( \hat{X} \) with compact support) we have \( \text{supp} f \subset \Lambda \) for some \( \Lambda \in \mathcal{O}_c(\hat{X}) \). We may identify any \( \hat{\gamma} \in \hat{\Gamma} \) with the positive integer-valued Radon measure \( \sum_{\hat{x} \in \hat{\gamma}} \epsilon_{\hat{x}} \), where \( \sum_{\hat{x} \in \emptyset} \epsilon_{\hat{x}} := 0 \) and define the pairing between \( f \in C_0(\hat{X}) \) and \( \hat{\gamma} \in \hat{\Gamma} \) by

\[
\langle f, \hat{\gamma} \rangle := \sum_{\hat{x} \in \hat{\gamma}} f(\hat{x}).
\]

As a result the Laplace transform is

\[
\int_{\hat{\Gamma}} \exp(\langle f, \hat{\gamma} \rangle) \pi_{z\sigma}(d\hat{\gamma}) = \exp\left[z \int_{\hat{X}} (e^{f(\hat{x})} - 1)\sigma(d\hat{x})\right].
\]

### 3.2 Marked Gibbs measures

We now describe a bigger class of probability measures on the configuration space \((\hat{\Gamma}, \mathcal{B}(\hat{\Gamma}))\), the so-called marked Gibbs measures. Our characterization results hold with respect to the so-called canonical marked Gibbs measures. In addition, we introduce also the grand canonical marked Gibbs measures which are more frequently used. The relation between these two classes is reviewed in Remark 3.3 and is known under the name equivalence of ensembles, cf. [Geo79], [Pre79]. Before, we need to introduce some standard concepts.

A symmetric measurable function \( V : \hat{X}^2 \to \mathbb{R} \cup \{+\infty\} \) is called a pair potential. For a given pair potential we define the energy functional \( E : \)
\( \hat{\Gamma}_0 \to \mathbb{R} \cup \{ +\infty \} \) by
\[
E(\hat{\eta}) := \sum_{(\hat{x}, \hat{y}) \in \eta} V(\hat{x}, \hat{y}),
\]
with \( E(\emptyset) := 0 \).

Let \( \hat{\eta} \in \hat{\Gamma}_0 \) and \( \hat{\gamma} \in \hat{\Gamma} \) be given, then the interaction energy between \( \hat{\eta} \) and \( \hat{\gamma} \) is given by
\[
W(\hat{\eta}, \hat{\gamma}) := \begin{cases} 
\sum_{\hat{x} \in \hat{\eta}, \hat{y} \in \hat{\gamma}} V(\hat{x}, \hat{y}), & \text{if } \sum_{\hat{x} \in \hat{\eta}, \hat{y} \in \hat{\gamma}} |V(\hat{x}, \hat{y})| < \infty \\
+\infty, & \text{otherwise}
\end{cases}
\]
For any \( \Lambda \in \mathcal{L}(\hat{X}) \) the conditional energy \( E_{\Lambda} : \hat{\Gamma} \to \mathbb{R} \cup \{ +\infty \} \) is given by
\[
E_{\Lambda}(\hat{\gamma}) := E(\hat{\gamma}_\Lambda) + W(\hat{\gamma}_\Lambda, \hat{\gamma}_{\hat{X} \setminus \Lambda}).
\]
Notice that the energy \( E \) may be expressed for any \( \hat{\gamma}, \hat{\gamma}' \in \hat{\Gamma}_0 \setminus \{ \emptyset \} \) with \( \hat{\gamma} \cap \hat{\gamma}' = \emptyset \) as
\[
E(\hat{\gamma} \cup \hat{\gamma}') = E(\hat{\gamma}) + E(\hat{\gamma}') + W(\hat{\gamma}, \hat{\gamma}').
\]

**Canonical marked Gibbs measures**

Now we are ready to define the specification belonging to the canonical system.

**Definition 3.2** The canonical specification \( \Pi_{\Lambda}^c, \Lambda \in \mathcal{B}_c(\hat{X}) \) is defined for any \( \hat{\gamma} \in \hat{\Gamma} \) and \( F \in \mathcal{B}(\hat{\Gamma}) \) by (cf. [Pre79])
\[
\Pi_{\Lambda}(F, \hat{\gamma}) := \frac{\mathbb{1} \{ 0 < Z_{\Lambda} < \infty \}(\hat{\gamma})}{Z_{\Lambda}(\hat{\gamma})} \int_{\Gamma(\hat{\gamma}_{\Lambda})} \mathbb{1}_F(\hat{\eta} \cup \hat{\gamma}_{\Lambda^c}) e^{-E(\hat{\eta}) - W(\hat{\eta}, \hat{\gamma}_\Lambda)} \sigma_{\hat{\gamma}_{\Lambda}^{-1}}(d\hat{\eta})
\]
and \( Z_{\Lambda}(\hat{\gamma}) := \int_{\Gamma(\hat{\gamma}_{\Lambda})} e^{-E(\hat{\eta}) - W(\hat{\eta}, \hat{\gamma}_\Lambda)} \sigma_{\hat{\gamma}_{\Lambda}^{-1}}(d\hat{\eta}) \). A probability measure \( \mu \) on \((\hat{\Gamma}, \mathcal{B}(\hat{\Gamma}))\) is called a canonical Gibbs measure if \( \mu \Pi_{\Lambda}^c = \mu, \forall \Lambda \in \mathcal{B}_c(\hat{X}) \) (the analogue of \((DLR)\)-equation for canonical Gibbs measures). We denote by \( G_{\Pi}^c(\sigma, V) \) the set of all such probability measures \( \mu \).

It has been shown in [Pre79] that, in fact, \((\Pi_{\Lambda}^c)_{\Lambda \in \mathcal{B}_c(\hat{X})} \) is a \((\mathcal{F}_{\Lambda})_{\Lambda \in \mathcal{B}_c(\hat{X})}\)-specification in the sense of [Föl75], where
\[
\mathcal{F}_{\Lambda^c} := \mathcal{F}_{\Lambda^c}(\Gamma) := \mathcal{B}_{\hat{X} \setminus \Lambda}(\Gamma) \vee \sigma(N_{\Lambda}^{-1}(N_0)).
\]
Our definition differs slightly from [Pre79], but for stable potentials fulfilling the conditions introduced in Subsection 5.1 they coincide. For all \( z > 0 \) the Poisson measure \( \pi \) belongs to \( \mathcal{G}_c(z\sigma, 0) \). The grand canonical Gibbs measures \( \mathcal{G}_{gc}(z, V) \) are given by the \((\mathcal{B}_\Lambda(\hat{\Gamma}))_{\Lambda \in \mathcal{B}_c(\hat{X})}\)-specification
\[
\Pi_{\Lambda}^{gc}(F, \hat{\gamma}) := \frac{1}{\Xi_{\Lambda}(\hat{\gamma})} \int_{\Gamma_\Lambda} 1 \mathbb{I}_{\{F, \hat{\gamma}\}} \mathbb{I}_{\{\hat{\gamma} \in \hat{\Gamma} \backslash \Lambda \cup \hat{\eta} \}} \mathbb{I}_{\{Z_{\Lambda}(\hat{\gamma}) < \infty \}} e^{-E_{\Lambda}(\hat{\gamma} \backslash (\Lambda \cup \hat{\eta}))} \pi_{\Lambda}(d\hat{\eta}).
\]

**Remark 3.3** Often one cannot work with the class of all Gibbs measures, but one has to restrict oneself to a subclass, for example the subclass defined by Assumption 4 below. Frequently, one assumes an a priori information about the support. Measures with this property we will call tempered in the following. In general one expect that the extremal canonical Gibbs measures are just extremal grand canonical Gibbs measure for a suitable value of \( z \), under an abstract condition this was proven in [Pre79]. This fact is called the equivalence of canonical and grand canonical ensemble. For the case \( X = \mathbb{R}^d \), \( \sigma = m \), and a continuous, finite range potential \( V \), such that there exists a decreasing function \( \psi : \mathbb{R}^+ \to \mathbb{R}^+_0 \) with \( V(x, y) \geq \psi(|x - y|) \) and \( \lim_{r \to 0} \psi(r)r^d = \infty \), H.-O. Georgii showed in [Geo76] the equivalence of ensembles for Gibbs measures tempered in the following sense:
\[
\mu \left( \left\{ \gamma \in \Gamma \mid \limsup_{k \to \infty} \frac{N_{\Lambda_k}(\gamma)}{\sigma(\Lambda_k)} < \infty \right\} \right) = 1,
\]
where \( \Lambda_k \) are the cubes centered at 0 of side length \( 2k \), \( k \in \mathbb{N} \). \( \Gamma \) denotes the space of configurations on \( \mathbb{R}^d \). The Gibbs measures tempered in this sense are a face of the class of all Gibbs measures.

For all \( \Lambda \in \mathcal{L}(\hat{X}) \) and all \( \mu \in \mathcal{G}_c(\sigma, V) \) we have
\[
\mu(\{ \hat{\gamma} \in \hat{\Gamma} \mid \Xi_{\Lambda}(\hat{\gamma}) < \infty, 0 < Z_{\Lambda}(\hat{\gamma}) < \infty \}) = 1.
\]

In fact, since for any \( \hat{\gamma} \in \hat{\Gamma} \) and \( \Lambda \in \mathcal{L}(\hat{X}) \) we have \( Z_{\Lambda}(\hat{\gamma}) \leq \Xi_{\Lambda}(\hat{\gamma}) \) and \( e^{-\sigma(\Lambda)} \leq \Xi_{\Lambda}(\hat{\gamma}) \), it is sufficient to estimate the measure of the following set
\[
A := \{ \hat{\gamma} \in \hat{\Gamma} \mid \Xi_{\Lambda}(\hat{\gamma}) = \infty \text{ or } Z_{\Lambda}(\hat{\gamma}) = 0 \}.
\]

But we have
\[
\mu(A) = \int_{\hat{\Gamma}} \mathbb{I}_{\Lambda}(\hat{\gamma}) \mathbb{E}_{\mu}(1|\mathcal{F}_{\Lambda^c})(\hat{\gamma}) \mu(d\hat{\gamma})
\]

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\[
\leq \int_\hat{\Gamma} (\mathbb{I}_{\{\hat{\xi}_A = \infty\}}(\hat{\gamma}) + \mathbb{I}_{\{\hat{\xi}_A > 0\}}(\hat{\gamma})) \Pi^c_A(\hat{\Gamma}, \hat{\gamma}) \mu(d\hat{\gamma})
\]

\[
\leq \int_\hat{\Gamma} (\mathbb{I}_{\{\hat{\xi}_A = \infty\}}(\hat{\gamma}) + \mathbb{I}_{\{\hat{\xi}_A > 0\}}(\hat{\gamma})) \frac{K_{\hat{\xi}_A, \hat{\zeta}_A}(\hat{\gamma})}{Z_A(\hat{\gamma})} \int_{\Gamma_A(\hat{\xi}_A, \hat{\gamma})} e^{-E(\hat{\eta}) - W(\hat{\eta}; \hat{\gamma})} \eta^\Lambda_{\sigma}(d\hat{\eta}) \mu(d\hat{\gamma})
\]

\[= 0,
\]

where \(K_{\hat{\xi}_A, \hat{\zeta}_A}(\hat{\gamma}) := \mathbb{I}_{\{\hat{\xi}_A < \infty\}}(\hat{\gamma}) \mathbb{I}_{\{\hat{\xi}_A < \infty\}}(\hat{\gamma})\). Moreover, according to Definition 3.2 we have

\[
\Pi^c_A(\{\hat{\eta} | \hat{\eta} \in \hat{\gamma}| \Lambda = \Lambda(\hat{\gamma})\}, \hat{\gamma}) = K_{\hat{\xi}_A, \hat{\zeta}_A}(\hat{\gamma})
\]

and therefore

\[
\mu(\{\hat{\gamma} \in \hat{\Gamma} | \Pi^c_A(\{\hat{\eta} | \hat{\eta} \in \hat{\gamma}| \Lambda = \Lambda(\hat{\gamma})\}, \hat{\gamma}) = 0\}) = 0.
\]

It follows from the properties of the specification \(\Pi^c_A\) that a probability measure \(\mu\) on \((\hat{\Gamma}, B(\hat{\Gamma}))\) is a canonical marked Gibbs measure iff for all \(\Lambda \in \mathcal{L}(\hat{X})\) and all \(Y \in B(\Gamma)\) there is a version of the conditional expectation such that

\[
\mathbb{E}_\mu[\mathbb{I}_Y | \mathcal{F}_{\Lambda^c}(\hat{\Gamma})] = \Pi^c_A(Y, \cdot), \ \mu - a.e.,
\]

where for a sub-\(\sigma\)-algebra \(\Sigma \subset B(\hat{\Gamma})\), \(\mathbb{E}_\mu[\cdot | \Sigma]\) denotes the conditional expectation with respect to \(\mu\) given \(\Sigma\).

We now prove some useful properties of canonical marked Gibbs measures which we need later on, namely in Section 5.

**Lemma 3.4** Let \(\mu\) be a probability measure on \((\hat{\Gamma}, B(\hat{\Gamma}))\), then for any \(B(\hat{\Gamma})\)-measurable function \(F\) which is either positive or \(\mu\)-integrable and any for \(\Lambda \in \mathcal{L}(\hat{X})\) there exists a probability kernel \(\mu_\Lambda : B(\hat{\Gamma}) \times \hat{\Gamma} \rightarrow \mathbb{R}^+\) such that

\[
\mathbb{E}_\mu(F | \mathcal{F}_{\Lambda^c})(\hat{\gamma}) = \int_{\Gamma_A(\hat{\xi}_A, \hat{\gamma})} F(\hat{\eta} \cup \hat{\zeta}_{\Lambda \setminus A}) \mu_\Lambda(d\hat{\eta}, \hat{\gamma}), \ \mu - a.s. \quad (3.22)
\]

Moreover for all \(F \in B(\hat{\Gamma})\) the function \(\mu_\Lambda(F, \cdot)\) is \(\mathcal{F}_{\Lambda^c}\)-measurable.

**Proof.** For any \(\Lambda \in \mathcal{L}(\hat{X})\) fixed, we define a measurable mapping \(p\) by

\[
p : \hat{\Gamma} \rightarrow \Gamma_{\Lambda^c} \times \mathbb{N}_0, \ \hat{\gamma} \mapsto (\hat{\gamma}_{\Lambda^c}, |\hat{\gamma}_A|).
\]

It is not hard to see that \(p\) is chosen in such a way that \(p^{-1}(B(\Gamma_{\Lambda^c} \times \mathbb{N}_0)) = \mathcal{F}_A\). Since \((\hat{\Gamma}, B(\hat{\Gamma}))\) is a Polish space, according to Parthasarathy [Par67, Chapter V, Theorem 8.1] there exists a regular conditional probability distribution
$P : \mathcal{B}(\hat{\Gamma}) \times (\Gamma_{c} \times \mathbb{N}_{0}) \to \mathbb{R}^{+}$ given $p$, with the following two properties: for all $B \in \mathcal{B}(\Gamma_{c} \times \mathbb{N}_{0}), \ D \in \mathcal{B}(\hat{\Gamma})$ and $\hat{\gamma} \in \hat{\Gamma}$ we have

$$\int_{p^{-1}(B)} P(D, p(\hat{\gamma})) = \int_{D \cap p^{-1}(B)} \mu(d\hat{\gamma}), \quad (3.23)$$

and

$$P(\hat{\Gamma} \setminus p^{-1}(\{\hat{\gamma}_{c}, N_{A}(\hat{\gamma})\}), p(\hat{\gamma})) = 0. \quad (3.24)$$

The regular conditional probability distribution $P$ is unique in the following sense: if we have two versions $P_{1}$ and $P_{2}$, then the set \{$(\hat{\gamma}, n) \in \Gamma_{c} \times \mathbb{N}_{0} | P_{1}(\cdot, (\hat{\gamma}, n)) \neq P_{2}(\cdot, (\hat{\gamma}, n))$\} has $p^*P_{1}$-measure zero.

Thus for all positive $\mathcal{B}(\hat{\Gamma})$-measurable function $F$ $\mu$-a.s. we have

$$\mathbb{E}_{\mu}(F|p^{-1}(\mathcal{B}(\Gamma_{c} \times \mathbb{N}_{0}))(\hat{\gamma})) = \int_{p^{-1}(p(\hat{\gamma}))} F(\hat{\gamma}') P(d\hat{\gamma}', p(\hat{\gamma})), \quad (3.25)$$

where

$$p^{-1}(\{p(\hat{\gamma})\}) = p^{-1}(\{(\hat{\gamma} \cap \Lambda_{c}, N_{A}(\hat{\gamma}))\})$$

$$= \{\hat{\gamma} \in \hat{\Gamma} | \hat{\gamma} \cap \Lambda_{c} = \hat{\gamma} \cap \Lambda_{c} \text{ and } N_{A}(\hat{\gamma}) = N_{A}(\hat{\gamma})\}$$

$$= \{\hat{\eta} \cup \hat{\gamma}_{c} | \hat{\eta} \in \Gamma_{c} \text{ and } N_{A}(\hat{\eta}) = N_{A}(\hat{\gamma})\}. \quad (3.24)$$

In fact, to derive (3.25) from (3.23) we may take $B' = p^{-1}(B)$ and use on the one hand the definition of conditional expectation to obtain

$$\int_{\hat{\Gamma}} \mathbb{1}_{B'}(\hat{\gamma}) F(\hat{\gamma}) \mu(d\hat{\gamma}) = \int_{B'} \mathbb{E}_{\mu}(F|p^{-1}(\mathcal{B}(\Gamma_{c} \times \mathbb{N}_{0}))(\hat{\gamma})) \mu(d\hat{\gamma}).$$

On the other hand, by the definition of the regular conditional probability distribution given $p$ we obtain that

$$\int_{\hat{\Gamma}} \mathbb{1}_{B'}(\hat{\gamma}) F(\hat{\gamma}) \mu(d\hat{\gamma}) = \int_{B'} \int_{\hat{\Gamma}} F(\hat{\gamma}') P(d\hat{\gamma}', p(\hat{\gamma})) \mu(d\hat{\gamma}).$$

Hence $\mu$-a.s.

$$\mathbb{E}_{\mu}(F|p^{-1}(\mathcal{B}(\Gamma_{c} \times \mathbb{N}_{0}))(\hat{\gamma})) = \int_{\hat{\Gamma}} F(\hat{\gamma}') P(d\hat{\gamma}', p(\hat{\gamma})).$$

Equality (3.25) follows taking into account the support property of the kernel $P$ in (3.24).
We now define the desired kernel \( \mu_\Lambda \) by
\[
\mu_\Lambda : \mathcal{B}(\Gamma_\Lambda) \times \hat{\Gamma} \to \mathbb{R}^+,
(A, \hat{\gamma}) \mapsto \mu_\Lambda(A, \hat{\gamma}) := P(p_\Lambda^{-1}(A), p(\hat{\gamma}))
\]
where \( p_\Lambda \) is given by (2.10). The result now follows from the equality (3.25), because without lost of generality we may take \( F \) of the form \( f = F_1 \circ p_\Lambda \cdot F_2 \circ p_\Lambda \) where \( F_1 \) is \( \mathcal{B}(\Gamma_\Lambda) \)-measurable and \( F_2 \) is \( \mathcal{B}_{\Lambda^c}(\hat{\Gamma}) \)-measurable. Then we have using (3.25)
\[
\mathbb{E}_\mu(F|\mathcal{F}_\Lambda)(\hat{\gamma}) = F_2(\hat{\gamma}_{\Lambda^c}) \int_{\mathbb{R}^{\mathcal{B}(\hat{\Gamma})}} F_1(\hat{\eta}) P(d\hat{\eta}, p(\hat{\gamma}))
\]
\[
= F_2(\hat{\gamma}_{\Lambda^c}) \int_{\hat{\Gamma}} F_1(\hat{\eta}) \mathbb{1}_{\{N_\Lambda = N_\Lambda(\hat{\eta})\}}(\hat{\eta}) P(d\hat{\eta}, p(\hat{\gamma}))
\]
\[
= \int_{\mathbb{R}^{\mathcal{B}(\hat{\Gamma})}} F_2(\hat{\gamma}_{\Lambda^c}) F_1(d\hat{\eta}) \mu_\Lambda(d\hat{\eta}, \hat{\gamma}).
\]

**Corollary 3.5** Let \( \mu \) be a probability measure on \((\hat{\Gamma}, \mathcal{B}(\hat{\Gamma}))\). Then for any \( \Lambda \in \mathcal{L}(\hat{X}) \)
\[
\int_{\mathbb{R}^{\mathcal{B}(\hat{\Gamma})}} F(\hat{\eta} \cup \hat{\gamma}_{\Lambda^c}) \mu_\Lambda(d\hat{\eta}, \hat{\gamma}) = \int_{\hat{\Gamma}} F(\hat{\eta}) \Pi_\Lambda(d\hat{\eta}, \hat{\gamma}), \mu - \text{a.s.}
\]
iff \( \mu \in \mathcal{G}_c(\sigma, V) \).

**Proof.** The result follows by (3.22) and the definition of canonical Gibbs measure.

Before we proceed we need the following definition.

**Definition 3.6** We call a probability measure \( \mu \) on \((\hat{\Gamma}, \mathcal{B}(\hat{\Gamma}))\) locally absolutely continuous with respect to \( \pi_\sigma \) iff \( \mu^\Lambda := \mu \circ p_\Lambda^{-1} \) is absolutely continuous with respect to \( \pi_\sigma^\Lambda := \pi_\sigma \circ p_\Lambda^{-1} \) for each \( \Lambda \in \mathcal{L}(\hat{X}) \).

**Lemma 3.7** Let \( \mu \in \mathcal{G}_c(\sigma, V) \) and \( \Lambda \in \mathcal{L}(\hat{X}) \) be given. Then \( \mu \) is locally absolutely continuous with respect to \( \pi_\sigma \) and
\[
\frac{d\mu^\Lambda}{d\pi_\sigma^\Lambda}(\hat{\eta}) := e^{-E(\hat{\eta})} \int_{\hat{\Gamma}} \frac{1}{Z_\Lambda(\hat{\gamma})} \mathbb{1}_{\{N_\Lambda = N_\Lambda(\hat{\eta})\}}(\hat{\gamma}) e^{-W(\hat{\eta}, \hat{\gamma}_{\Lambda^c})} \mu(d\hat{\gamma}). \tag{3.26}
\]
Proof. Let $F$ be a positive $\mathcal{B}(\Gamma_\Lambda)$-measurable function. By a direct computation we see that

$$
\int_{\hat{\Gamma}} F(\hat{\gamma}) \mu(d\hat{\gamma}) = \int_{\hat{\Gamma}} \mathbb{E}_\mu[F \circ p_\Lambda|\mathcal{F}_\Lambda](\hat{\gamma}) \mu(d\hat{\gamma})
$$

$$
= \int_{\hat{\Gamma}} \int_{\Gamma_\Lambda} F(\hat{\eta}) \Pi^\Lambda_\sigma(d\hat{\eta}, \hat{\gamma}) \mu(d\hat{\gamma})
$$

$$
= \int_{\Gamma_\Lambda} \int_{\hat{\Gamma}} 1_{\{N_\Lambda = N_\Lambda(\hat{\eta})\}}(\hat{\gamma}) F(\hat{\eta}) \frac{K_{\hat{\Xi}_\Lambda, \hat{\Lambda}}(\hat{\gamma})}{Z_{\Lambda}(\hat{\gamma})} e^{-E(\hat{\eta}) - W(\hat{\eta}, \hat{\gamma})} \mu(d\hat{\gamma})
$$

$$
= \int_{\Gamma_\Lambda} F(\hat{\eta}) \left[ e^{-E(\hat{\eta})} \int_{\hat{\Gamma}} 1_{\{N_\Lambda = N_\Lambda(\hat{\eta})\}}(\hat{\gamma}) \frac{K_{\hat{\Xi}_\Lambda, \hat{\Lambda}}(\hat{\gamma})}{Z_{\Lambda}(\hat{\gamma})} e^{-W(\hat{\eta}, \hat{\gamma})} \mu(d\hat{\gamma}) \right] \mu(d\hat{\eta}).
$$

On the other hand, it follows from the calculations after Remark 3.3 that $1_{\{\hat{\Xi}_\Lambda < \infty\}}(\hat{\gamma}) 1_{\{Z_{\Lambda} > 0\}}(\hat{\gamma}) = 1$ $\mu$-a.s., and this implies (3.26) as desired. 

3.3 The $K$-transform

In this subsection we introduce an useful transformation between functions on $\hat{\Gamma}_0$ and $\hat{\Gamma}$.

We denote by $L^0(\hat{\Gamma}_0) := L^0(\hat{\Gamma}_0, \mathcal{B}(\hat{\Gamma}_0))$ the set of all measurable functions $F : \hat{\Gamma}_0 \rightarrow \mathbb{R}$. $B(\hat{\Gamma}_0) \subset L^0(\hat{\Gamma}_0)$ is the subset of all bounded measurable functions. $L^0_{ls}(\hat{\Gamma}_0)$ denotes the set of all measurable functions with local support, i.e., $G \in L^0_{ls}(\hat{\Gamma}_0)$ iff $G \in L^0(\hat{\Gamma}_0)$ and there exists a $\Lambda \in \mathcal{L}(\hat{X})$ such that $G |_{\hat{\Gamma}_0 \setminus \Gamma_{\Lambda}} = 0$. The set of all measurable functions with bounded support is denoted by $L^0_{bs}(\hat{\Gamma}_0)$, i.e., $G \in L^0_{bs}(\hat{\Gamma}_0)$ iff there exist $\Lambda \in \mathcal{L}(\hat{X})$ and $N \in \mathbb{N}$ such that $G |_{\hat{\Gamma}_0 \setminus (\bigcup_{n=0}^N \Gamma^n_\Lambda)} = 0$. We define the sets $B_{ls}(\hat{\Gamma}_0)$ and $B_{bs}(\hat{\Gamma}_0)$, respectively of bounded measurable functions in a similar way.

On $\hat{\Gamma}$ we define the algebra of cylinder sets

$$
\mathcal{B}_{cyl}(\hat{\Gamma}) := \bigcup_{\Lambda \in \mathcal{L}(\hat{X})} \mathcal{B}_\Lambda(\hat{\Gamma}).
$$

A cylinder function $F : \hat{\Gamma} \rightarrow \mathbb{R}$ is a measurable function from $L^0(\hat{\Gamma}, \mathcal{B}(\hat{\Gamma}))$ which is also measurable with respect to $\mathcal{B}_\Lambda(\hat{\Gamma})$ for some $\Lambda \in \mathcal{L}(\hat{X})$ (we call $\Lambda$ a domain of cylindricity of $F$). We denote the set of these functions by
Using the \( \sigma \)-isomorphism between \( \mathcal{B}_\Lambda(\hat{\Gamma}) \) and \( \mathcal{B}(\Gamma_\Lambda) \) we have that \( F \in L^0(\hat{\Gamma}, \mathcal{B}_\Lambda(\hat{\Gamma})) \), if \( F \upharpoonright_{\Gamma_\Lambda} \in L^0(\Gamma_\Lambda, \mathcal{B}(\Gamma_\Lambda)) \) and we have the following connection

\[
F(\hat{\gamma}) = F \upharpoonright_{\Gamma_\Lambda} (\hat{\gamma}_\Lambda).
\]

A cylinder function \( F \) is polynomially bounded, in symbols \( F \in \mathcal{F}L^0_{pb}(\hat{\Gamma}, \mathcal{B}(\hat{\Gamma})) \), if \( F \) is measurable with respect to \( \mathcal{B}_\Lambda(\hat{\Gamma}) \) for some \( \Lambda \in \mathcal{L}(\hat{\Gamma}) \) and there exists a polynomial \( P \) on \( \mathbb{R} \) such that \( |F(\hat{\gamma}_\Lambda)| \leq P( |\hat{\gamma}_\Lambda| ) \). Let \( D := C^\infty_0(\hat{X}) \) be the set of all \( C^\infty \) functions on \( \hat{X} \) with compact support and define \( \mathcal{FC}^\infty_b(D, \hat{\Gamma}) \) as the set of all \( C^\infty \) cylinder functions \( F : \hat{\Gamma} \rightarrow \mathbb{R} \) of the form

\[
F(\hat{\gamma}) = g_F(\langle \varphi_1, \hat{\gamma} \rangle, \ldots, \langle \varphi_n, \hat{\gamma} \rangle), \quad \hat{\gamma} \in \hat{\Gamma},
\]

where \( \varphi_1, \ldots, \varphi_n \in D \) and \( g_F \in C^\infty_0(\mathbb{R}^n) \). Furthermore \( \mathcal{FC}^k_p(D, \hat{\Gamma}, \mathcal{B}_\Lambda(\hat{\Gamma})) \), \( k \in \mathbb{N} \) denotes the set of all \( k \) times differentiable cylinder functions which are polynomially bounded.

We are now ready to define the \( K \)-transform.

**Definition 3.8** Let \( G \in L^0(\hat{\Gamma}_0) \) be given. Then we define \( KG \) by

\[
KG : \hat{\Gamma} \rightarrow \mathbb{R}, \quad \hat{\gamma} \mapsto (KG)(\hat{\gamma}) := \sum_{\xi \in \hat{\gamma}} G(\hat{\xi}), \tag{3.27}
\]

where the sum is extended over all finite subconfigurations \( \hat{\xi} \) from \( \hat{\gamma} \), in symbols \( \hat{\xi} \in \hat{\gamma} \).

**Remark 3.9** The \( K \)-transform appears from different point of view in statistical mechanics, see e.g., [Bog46], and [KK02a] for details. In this paper we will use the \( K \)-transform as a tool to prove support properties of measures on the configuration space \( \hat{\Gamma} \) introduced above.

Let us collect some properties of the \( K \)-transform, for the proofs we refer to e.g., [Len75] and [KK02a].

**Proposition 3.10**

1. The \( K \)-transform maps \( L^0_{ls}(\hat{\Gamma}_0) \) into \( \mathcal{F}L^0(\hat{\Gamma}) \). In particular, if \( G \upharpoonright_{\hat{\Gamma}_0 \setminus \hat{\Gamma}_\Lambda} = 0 \) for some \( \Lambda \in \mathcal{L}(\hat{X}) \), then \( KG \in L^0(\hat{\Gamma}, \mathcal{B}_\Lambda(\hat{\Gamma})) \).

2. The \( K \)-transform maps \( B_{bs}(\hat{\Gamma}_0) \) into \( \mathcal{F}L^0_{pb}(\hat{\Gamma}) \), in particular if \( G \upharpoonright_{\hat{\Gamma}_0 \setminus \bigcup_{n=0}^{N} \hat{\Gamma}_0^{(n)}} = 0 \) for some \( \Lambda \in \mathcal{L}(\hat{X}) \) and \( N \in \mathbb{N} \), then there exists \( C > 0 \) such that

\[
|KG(\hat{\gamma})| \leq C(1 + |\hat{\gamma}_\Lambda|)^N.
\]
3. The mapping \( K : L^0_\text{i}(\hat{\Gamma}_0) \rightarrow \mathcal{F}L^0(\hat{\Gamma}) \) is invertible with inverse given by

\[
K^{-1}F(\hat{\eta}) := \sum_{\hat{\xi} \subset \hat{\eta}} (-1)^{|\hat{\eta} \setminus \hat{\xi}|} F(\hat{\xi}).
\]

3.4 Correlations measures

We now proceed with the definition of correlation measure on the finite configuration space \( \hat{\Gamma}_0 \), cf. [Len73]. For this we need to introduce the notion of measures on \((\hat{\Gamma}_0, \mathcal{B}(\hat{\Gamma}_0))\) which are locally finite.

**Definition 3.11**

1. A probability measure \( \mu \) on \((\hat{\Gamma}, \mathcal{B}(\hat{\Gamma}))\) has finite local moments iff for all \( \Lambda \in \mathcal{L}(\hat{X}) \) we have

\[
\int_{\hat{\Gamma}} |\hat{\gamma}_\Lambda|^n \mu(d\hat{\gamma}) < \infty, \quad \forall n \in \mathbb{N}_0.
\]

The set of all such measures is denoted by \( \mathcal{M}^1_{\text{fm}}(\hat{\Gamma}) \).

2. A measure \( \rho \) on \((\hat{\Gamma}_0, \mathcal{B}(\hat{\Gamma}_0))\) is locally finite iff for every \( \Lambda \in \mathcal{L}(\hat{X}) \) and every \( n \in \mathbb{N} \) it is \( \rho(\hat{\Gamma}_0^{(n)}) < \infty \). We denote the set of all such measures by \( \mathcal{M}_{\text{lf}}(\hat{\Gamma}_0) \).

**Definition 3.12** Let \( \mu \in \mathcal{M}^1_{\text{fm}}(\hat{\Gamma}) \) be given. Then the correlation measure \( \rho_\mu \) corresponding to \( \mu \) is uniquely defined by

\[
\rho_\mu(A) := \int_{\hat{\Gamma}} (K 1_A)(\hat{\gamma}) \mu(d\hat{\gamma}), \quad \forall A \in \mathcal{B}_0(\hat{\Gamma}_0).
\]

**Remark 3.13**

1. By general duality we may define the following transformation of measures

\[
K^* : \mathcal{M}^1_{\text{fm}}(\hat{\Gamma}) \rightarrow \mathcal{M}_{\text{lf}}(\hat{\Gamma}_0), \quad \mu \mapsto K^* \mu := \rho_\mu.
\]

2. Since \( \hat{\Gamma}_0 \) is the disjoint union of the family of measurable spaces \((\Gamma_0^{(n)}, \mathcal{B}(\Gamma_0^{(n)}))_{n \in \mathbb{N}_0} \) (cf. (2.6)), \( \rho_\mu \) is defined if we know each component \( \rho_\mu^{(n)} \), \( n \in \mathbb{N}_0 \). The \( \sigma \)-finite measure \( \rho_\mu^{(n)} \) on the corresponding measurable space \((\Gamma_0^{(n)}, \mathcal{B}(\Gamma_0^{(n)}))\) are called the \( n \)-th correlation measure.
3. The correlation measure corresponding to the marked Poisson measure is simply the marked Lebesgue-Poisson measure. Indeed, for any \( A \in B(\hat{\Gamma}_0) \) with \( A \subset \Gamma_\Lambda \) for some \( \Lambda \in \mathcal{L}(X) \) we have

\[
\rho_{\pi_\sigma}(A) = \int_{\Gamma}(K\mathbb{1}_A)(\hat{\gamma})\pi_\sigma(d\hat{\gamma}) = \int_{\Gamma_\Lambda}(K\mathbb{1}_A)(\hat{\eta})\pi_\sigma(\hat{\eta})
\]

\[
= e^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} (K\mathbb{1}_A)(\{\hat{x}_1, \ldots, \hat{x}_n\})\sigma(d\hat{x})_1^n
\]

\[
= e^{-\sigma(\Lambda)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\Lambda^n} \sum_{k=0}^{n} \binom{n}{k} \mathbb{1}_A(\{\hat{x}_1, \ldots, \hat{x}_n\})\sigma(d\hat{x})_1^n
\]

\[
= \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\Lambda^k} \mathbb{1}_A(\{\hat{x}_1, \ldots, \hat{x}_k\})\sigma(d\hat{x})_1^k
\]

\[
= \lambda_\sigma(A).
\]

**Corollary 3.14** Let \( \mu \in \mathcal{M}_{fm}(\hat{\Gamma}) \) be given. For all positive measurable functions \( G \in L^0(\hat{\Gamma}_0) \) we have

\[
\int_{\hat{\Gamma}_0} G(\hat{\eta})\rho_\mu(\hat{\eta}) = \int_{\hat{\Gamma}} (KG)(\hat{\gamma})\mu(\hat{\gamma}).
\] (3.28)

The following proposition gives us an explicit relation between the local density of \( \mu \), cf. (3.26) and the density of the associated correlation measure \( \rho_\mu \) with respect to the marked Lebesgue-Poisson measure. For the proof see e.g., [KK02a, Proposition 4.14].

**Proposition 3.15** Let \( \mu \in \mathcal{M}_{fm}(\hat{\Gamma}) \) be a measure which is locally absolutely continuous with respect to \( \pi_\sigma \). Then \( \rho_\mu := K^*\mu \) is absolutely continuous with respect to \( \lambda_\sigma \) and for all \( \Lambda \in \mathcal{L}(X) \) we have

\[
k_\mu(\hat{\eta}) := \frac{d\rho_\mu}{d\lambda_\sigma}(\hat{\eta}) = \int_{\Gamma_\Lambda} \frac{d\mu^\Lambda}{d\pi_\sigma^\Lambda}(\hat{\gamma} \cup \hat{\eta})\pi_\sigma^\Lambda(d\hat{\gamma}), \text{ for } \lambda_\sigma - \text{a.e. } \hat{\eta} \in \hat{\Gamma}_\Lambda.
\] (3.29)

To control several expressions in this work it is important to extend the \( K \)-transform to functions from \( L^1(\hat{\Gamma}_0, \rho_\mu) \), cf. Subsection 5.1.

**Theorem 3.16** (cf. [KK02a, Theorem 4.11]) Let \( \mu \in \mathcal{M}_{fm}(\hat{\Gamma}) \) and \( \rho_\mu \) the corresponding correlation measure. For any \( G \in L^1(\hat{\Gamma}_0, \rho_\mu) \) the series

\[
\sum_{\hat{\eta} \in \hat{\gamma}} G(\hat{\eta}),
\] (3.30)
is \(\mu\)-a.s. absolutely convergent. We define \((KG)(\hat{\gamma}) := \sum_{\hat{\eta} \in \hat{\gamma}} G(\hat{\eta})\). In addition, since

\[
|KG|_{L^1(\mu)} \leq |K|G|_{L^1(\mu)} = |G|_{L^1(\rho_\mu)},
\]

we have \(KG \in L^1(\hat{\Gamma}, \mu)\) and

\[
\int_{\Gamma_0} G(\hat{\eta})\rho_\mu(d\hat{\eta}) = \int_{\hat{\Gamma}} (KG)(\hat{\gamma})\mu(d\hat{\gamma}).
\]

### 3.5 Measures on marked configuration spaces

Actually the measures introduced before, as measures on \(\hat{\Gamma}\), have already full measure with respect to the marked configuration space \(\Omega\), see (3.33) for its definition. In this subsection we present a simple and clear proof for this part using as a tool the \(K\)-transform.

**Lemma 3.17** Let \(\mu \in \mathcal{M}^{1}_{fm}(\hat{\Gamma})\) be given. Consider the set

\[
A := \{((x, t), (x, s)) \in \hat{X}^2, t, s \in S\}
\]

and suppose that \(\rho_\mu^{(2)}(A) = 0\), then

\[
\Omega = \Omega_X(S) := \{\hat{\gamma} := \{(x, s_x) | x \in \gamma \} \in \hat{\Gamma} | \gamma \in \Gamma_X, s_x \in S, \forall x \in \gamma\} \quad (3.33)
\]

the set of marked configurations has full \(\mu\)-measure.

**Proof.** Notice that

\[
\Omega_E := \{\hat{\gamma} \in \hat{\Gamma} | \exists x \in X, s, t \in S, t \neq s \text{ and } (x, t), (x, s) \in \hat{\gamma}\} \quad (3.34)
\]

and hence \((K\mathbb{1}_A)(\hat{\gamma}) \geq \mathbb{1}_{\Omega_E}(\hat{\gamma})\) according to (3.28) we have

\[
\mu(B) \leq \int_{\Gamma} (K\mathbb{1}_A)(\hat{\gamma})\mu(d\hat{\gamma}) = \int_{\hat{X}^2} \mathbb{1}_A(\hat{x}, \hat{y})\rho_\mu^{(2)}(d\hat{x}, d\hat{y}) = \rho_\mu^{(2)}(A) = 0.
\]

**Remark 3.18** The above lemma shows that the measure \(\mu\) can be considered as a measure on \(\Omega_X(S)\) (the set of marked configurations). More precisely, the set of all configurations \(\hat{\gamma} \in \hat{\Gamma}\) such that if \((x, s) \in \hat{\gamma}\), then
Corollary 3.19. Canonical marked Gibbs measures $\mu \in \mathcal{M}^1_{fm}(\hat{\Gamma})$ may be considered as measures on the marked configuration space.

**Proof.** In fact, Lemma 3.7 and Proposition 3.15 implies that $\rho_{\mu}^{(2)}$ is absolutely continuous with respect to $\sigma_2$. Since $\sigma$ is non-atomic, i.e., $\sigma(\{\hat{x}\}) = 0$ for all $\hat{x} \in \hat{X}$, then the result follows by Lemma 3.17.

Remark 3.20. As the property to be a marked configuration is already determined by all pairs of points in the configuration, naturally the second correlation measure appears. The above scheme may be applied to similar situations. For example, let $\mu$ be a point random field (cf. [Kal83] for this notion), then several particles may have the same position. However, if the diagonal in $\hat{X}^2$ has measure zero with respect to the second correlation measure $\rho_{\mu}^{(2)}$, then $\mu$ has already full measure on the configuration space itself. Such random fields are called simple, for more details cf. e.g., [DVJ88] and [KK02b].
4 Characterization of marked Poisson measures

4.1 Diffeomorphisms groups and vector fields

In this subsection we describe different kinds of diffeomorphisms groups and spaces of vector fields which will be used below.

Let us denote the group of all diffeomorphisms \( \hat{\phi} : \hat{X} \to \hat{X} \) by \( \text{Diff}(\hat{X}) \) and by \( \text{Diff}_0(\hat{X}) \) the subgroup of all diffeomorphisms \( \hat{\phi} \) which are equal to the identity function outside of a compact set. The role of the corresponding algebra is played by \( \text{Vect}(\hat{X}) \), the set of all vector fields \( \hat{v} : \hat{X} \to T\hat{X} \) on \( \hat{X} \) and respectively by \( \text{Vect}_0(\hat{X}) \), the subset of all vector fields \( \hat{v} \in \text{Vect}(\hat{X}) \) with compact support. On the one hand we are interested in large subgroups of \( \text{Diff}(\hat{X}) \), subspace of \( \text{Vect}(\hat{X}) \), respectively such that the canonical marked Gibbs measures on \( \hat{\Gamma} \) are quasi-invariant, allow an integration by parts formula, respectively. A large subgroup for which our technique applies easily is

\[
\text{Diff}_{\text{large}}(\hat{X}) := \{ \hat{\phi} \in \text{Diff}(\hat{X}) | \text{supp} \hat{\phi} \subset \Lambda \times S \text{ for some } \Lambda \in \mathcal{B}_c(\hat{X}) \}
\]

and the corresponding algebra

\[
\text{Vect}_{\text{large}}(\hat{X}) := \{ \hat{v} \in \text{Vect}(\hat{X}) | \text{supp} \hat{v} \subset A \times S, A \in \mathcal{B}_c(\hat{X}), \sup_{\hat{x} \in \hat{X}} (|\text{div}^\hat{X} \hat{v}(\hat{x})| + |\hat{v}(\hat{x})|_{T\hat{X}}) < \infty \}.
\]

Here \( \text{div}^\hat{X} \) is the divergence in \( \hat{X} \) with respect to \( \hat{m} \). The support property is sufficient in order to apply the theory of specification.

On the other hand we would like to characterize measures on \( \hat{\Gamma} \) via their Radon-Nikodym derivative or their integration by parts formula. In this case it is interesting to search for a small subgroup \( \text{Diff}_{\text{small}}(\hat{X}) \) of \( \text{Diff}(\hat{X}) \), small subspace \( \text{Vect}_{\text{small}}(\hat{X}) \) of \( \text{Vect}(\hat{X}) \) respectively. There exists no natural choice for this small subgroup, it depends on the application one have in mind. This is discussed concretely in more details in Example 4.3 and in the applications in Section 8. The subgroup \( \text{Diff}_{\text{small}}(\hat{X}) \subset \text{Diff}_{\text{large}}(\hat{X}) \) need to fulfill the following properties. It is a subgroup algebraically generated by a countable family of locally compact groups. We denote by \( \text{Diff}_{\text{small}}(O) \) the subgroup of all \( \hat{\phi} \in \text{Diff}_{\text{small}}(\hat{X}) \) with support in \( O \). Furthermore, we assume that for each \( O \in \mathcal{O}_c(X) \) the group \( \text{Diff}_{\text{small}}(O) \) characterize measures on \( O \) by their Radon-Nikodym derivative, i.e., if a measure \( \tilde{\sigma} \) on \( O \) has the same
Radon-Nikodym derivative as $\sigma$ then there exists a constant $k > 0$ such that $\tilde{\sigma} = k \sigma$.

For $\text{Vect}_{\text{small}}(\hat{X})$ we may take a countable set in $\text{Vect}_0(\hat{X})$ which is closed under addition and for every open set $O \in \mathcal{O}_c(\hat{X})$ we have an integration by parts characterization, i.e., let $\tilde{\sigma}$ be a measure on $O$ such that for all $\hat{v} \in \text{Vect}_{\text{small}}(\hat{X})$ with $\text{supp} \hat{v} \subset O$ and all $F \in C^\infty_0(O)$ we have

$$\int_O \langle \nabla^{\hat{X}} F(\hat{x}), \hat{v}(\hat{x}) \rangle_{T\hat{x}^{\hat{X}}} d\tilde{\sigma}(d\hat{x}) = -\int_O F(\hat{x}) \text{div}^{\hat{X}} \hat{v}(\hat{x}) \tilde{\sigma}(d\hat{x}) + \int_O F(\hat{x})(\hat{v}(\hat{x}), \beta_{\tilde{\sigma}}(\hat{x}))_{T\hat{x}^{\hat{X}}} d\tilde{\sigma}(d\hat{x}),$$

and $\beta_{\tilde{\sigma}} \in L_{\text{loc}}^1(O, m)$. Then there exists $k > 0$ such that $\tilde{\sigma}(d\hat{x}) = k \sigma(d\hat{x})$.

In the following example we construct $\text{Diff}_{\text{small}}(\hat{X})$ and $\text{Vect}_{\text{small}}(\hat{X})$ explicitly.

**Example 4.1** Let $X$ be a manifold with volume element $m$ which is a quasi-invariant measure with respect to $\text{Diff}_0(X)$. The construction of the group $\text{Diff}_{\text{small}}(X)$, given here, was already used by [GGV75] and [Shi94]. For convenience of the reader we repeat their ideas, because in the other examples we need to extend and modify in order to cover these richer models. Since $X$ is a manifold, it can be covered with countably many charts. Thus we may reduce ourself to open balls in $\mathbb{R}^d$. Without lost of generality $B_1(0) := \{x \in \mathbb{R}^d | |x| < 1\}$. For every $n \in \mathbb{N}$ consider $\chi_n \in C^\infty(\mathbb{R}^d)$ such that $\chi_n|_{B_{1-\frac{1}{n}}(0)} = 1$ and $\chi_n|_{B_1(0)} = 0$. Then for every unit vector $e_i \in \mathbb{R}^d$, $i = 1, \ldots, d$ we can construct the vector field $v(x)$, $x \in O$ by

$$v^{i,n}(x) = \chi_n(x)e_i$$

and the corresponding flow $\phi^{i,n}_t$. As a one-parameter group

$$\text{Diff}^{i,n}_{\text{small}}(O) := \{\phi^{i,n}_t | t \in \mathbb{R}\}$$

is locally compact. $\text{Diff}_{\text{small}}(X)$ we define as the group algebraically generated by $\text{Diff}^{i,n}_{\text{small}}(O)$ for $i, n \in \mathbb{N}$ and $O$ from a countable cover.

Let $\mu$ be a $\text{Diff}_{\text{small}}(O)$-quasi-invariant measure with Radon-Nikodym derivative of the form

$$\frac{d((\phi^{i,n}_t)^*\mu)}{d\mu}(x) = \frac{r((\phi^{i,n}_t)^{-1}(x))}{r(x)} \frac{d((\phi^{i,n}_t)^*m)}{dm}(x),$$

for a certain $r > 0$, $m$-a.s. Then there exists a constant $k > 0$ such that $\mu = km$. 21
Proof. Define $\hat{\mu} := \frac{1}{r} \mu$ and let $\phi \in \text{Diff}_{\text{small}}(O)$ be given. Then we have $d(\phi^* \hat{\mu}) = \det(D\phi) d\hat{\mu}$. Let $F$ be a smooth function on $O$ with $\text{supp} F$ compact in $O$. Then there exists $O_{n-1}$ such that $\text{supp} F \subset O_{n-1}$. Moreover for small enough $t$ one has $F(\phi_{i,n}^t(x)) = F(x + te_i)$. This can be seen from the following calculation

$$\frac{d}{dt} F(\phi_{i,n}^t(x)) = \langle \nabla F(\phi_{i,n}^t(x)), v^{i,n}(\phi_{i,n}^t(x)) \rangle$$

$$= \partial_i F(\phi_{i,n}^t(x)) \chi_{O_n}(\phi_{i,n}^t(x))$$

$$= \partial_i F(\phi_{i,n}^t(x)).$$

In the last equality we used the fact that given $x \in O_{n-1}$ there exists $T$ such that if $t < T$, $\phi_{i,n}^t(x) \in O_n$. Hence the quasi-invariant property of $\mu$ implies that if $i, n, t$ are as before we have

$$\int_O F(\phi_{i,n}^t(x)) \hat{\mu}(dx) = \int_O F(x + te_i) \hat{\mu}(dx).$$

On the other hand since $\det D\phi_{i,n}^t(x) = 1$ the left hand side of the above equality gives

$$\int_O F(\phi_{i,n}^t(x)) \hat{\mu}(dx) = \int_O F(x) \hat{\mu}(dx).$$

From this we conclude that for all $F \in C^\infty(O)$ the measure $\hat{\mu}$ is invariant for small translations in the directions $e_i$, $i = 1, \ldots, d$. Therefore, there exists $c > 0$ such that $\hat{\mu} = cm$. 

Example 4.2 Let $X$ be a $d$-dimensional and $S$ a $d'$-dimensional manifold with volume elements $m_X$ and $m_S$. Then $m = m_X \otimes m_S$ is a quasi-invariant measure with respect to $\text{Diff}_0(X \times S, \text{Diff}(S))$, where

$$\text{Diff}_0(X \times S, \text{Diff}(S)) := \{ \hat{\phi} : X \times S \to X \times S, \hat{\phi}(x, s) = (\phi(x), \psi(x, s)) \},$$

$\phi \in \text{Diff}_0(X)$ and the mapping $\psi$ is such that $\psi(x, \cdot) \in \text{Diff}(S)$ on $S$ for any fixed $x \in X$. The choice of the group $\text{Diff}_0(X \times S, \text{Diff}(S))$ is related to the fact that it leaves the fibre bundle structure of $X \times S$ invariant. In contrast to $\text{Diff}_0(X \times S)$ the group $\text{Diff}_0(X \times S, \text{Diff}(S))$ maps also marked configurations into marked configurations. Therefore, this group is suitable to treat marked point processes.
Since \( X \times S \) is a manifold, it can be covered with countably many charts. Thus we may reduce ourself to open balls in \( \mathbb{R}^{d+d'} \). Without lost of generality \( B_1(0) := \{ \hat{x} \in \mathbb{R}^{d+d'} \mid |\hat{x}| < 1 \} \). For every \( n \in \mathbb{N} \) consider \( \chi_n \in C^\infty(\mathbb{R}^{d+d'}) \) such that \( \chi_n \restriction B_{1-\frac{1}{n}}(0) = 1 \) and \( \chi_n \restriction B'_1(0) = 0 \). Then for every unit vector \( e_i \in \mathbb{R}^d, i = 1, \ldots, d+d' \) we can construct the vector field \( v(\hat{x}), \hat{x} \in O \) by
\[
v^{i,n}(\hat{x}) = \chi_n(\hat{x})e_i
\]
and the corresponding flow \( \hat{\phi}^{i,n}_t \). The following group of diffeomorphims in \( O \)
\[
\text{Diff}^{i,n}_{\text{small}}(O \times S, \text{Diff}(S)) := \{ \hat{\phi}^{i,n}_t \mid t \in \mathbb{R} \}
\]
is locally compact. \( \text{Diff}^{i,n}_{\text{small}}(X \times S, \text{Diff}(S)) \) we define as the group algebraically generated by \( \text{Diff}^{i,n}_{\text{small}}(O \times S, \text{Diff}(S)) \) for \( i, n \in \mathbb{N} \) and \( O \) from a countable cover. Working with the manifold \( X \times S \) instead of \( X \) we obtain following the argumentation in Example 4.1 we obtain the following result.

Let \( \mu \) be a \( \text{Diff}^{i,n}_{\text{small}}(O \times S, \text{Diff}(S)) \)-quasi-invariant measure with Radon-Nikodym derivative of the form
\[
d((\hat{\phi}^{i,n}_t)_*\mu) (x,s) = r((\hat{\phi}^{i,n}_t)^{-1}(x,s)) \frac{d((\hat{\phi}^{i,n}_t)_*\mu)}{dm}(x,s),
\]
for a certain \( r > 0 \), \( m \)-a.s. Then there exists a constant \( k > 0 \) such that
\[
\mu = kr \mu.
\]

**Example 4.3** Let \( X \) be a manifold with volume element \( m_X \), \( S \) a homogeneous space with respect to a locally compact group \( G \) and \( m_S \) a \( G \) quasi-invariant measure on \( S \). Then \( m = m_X \otimes m_S \) is a quasi-invariant measure with respect to \( \text{Diff}_0(X \times S, G) \), where
\[
\text{Diff}_0(X \times S, G) := \{ \hat{\phi} : X \times S \to X \times S, \hat{\phi}(x,s) = (\phi(x), \psi(x,s)) \},
\]
\( \phi \in \text{Diff}_0(X) \) and the mapping \( \psi \) is such that for any fixed \( x \in X \) there exists \( g_x \in G \) such that \( \psi(x, \cdot) = a(g_x \cdot) \), where \( a : G \times S \to S \) is the action of \( G \) in \( S \). Notice that we may lift the action \( a \) from \( S \) to \( X \times S \). In fact, we define \( a(g, (x,s)) := (x, a(g,s)) \). Since \( X \) is a manifold, we can construct the vector-fields \( v^{i,n}_t \) and the corresponding flow \( \hat{\phi}^{i,n}_t \) as in Example 4.1. Given \( g \in G \) we define a mapping from \( X \times S \) to itself by
\[
\hat{\phi}^{i,n}_t(g, (x,s)) := (\phi^{i,n}_t(x), a(g,s)).
\]
The following group of diffeomorphisms in $O$

$$\text{Diff}_{\text{small}}(O \times S, G) := \{ \hat{\phi}^{i,n,g}_{t} \mid t \in \mathbb{R}, g \in G \}$$

is locally compact. $\text{Diff}_{\text{small}}(X \times S, G)$ we define as the group algebraically generated by $\text{Diff}_{\text{small}}(O \times S)$ for $i, n \in \mathbb{N}$ and $O$ from a countable cover.

Let $\mu$ be a $\text{Diff}_{\text{small}}(O \times S, G)$-quasi-invariant measure with Radon-Nikodym derivative of the form

$$d((\hat{\phi}^{i,n,g}_{t})^* \mu) = \frac{r((\hat{\phi}^{i,n,g}_{t})^{-1}(x,s))}{r(x,s)} \frac{d((\hat{\phi}^{i,n,g}_{t})^* m)}{dm}(x,s),$$

for a certain $r > 0$, $m$-a.s. Then there exists a constant $k > 0$ such that $\mu = km$.

**Proof.** Define $\tilde{\mu} := \frac{1}{r} \mu$ and let $\hat{\phi} \in \text{Diff}_{\text{small}}(O \times S)$ be given. Then we have $d(\hat{\phi}^* \tilde{\mu}) = \det(D\hat{\phi})d\tilde{\mu}$. The projection of $\tilde{\mu}$ on $X$ we denote by $\tilde{\mu}^X$, i.e., for any $A \in \mathcal{B}(O)$

$$\tilde{\mu}^X(A) := \tilde{\mu}(A \times S).$$

If $e$ denotes the identity of $G$, then since $\hat{\phi}^{i,n,e}_{t}(x,s) = (\phi^{i,n}_{t}(x), s)$ we have $\det(D\hat{\phi}^{i,n,e}_{t})(x,s) = \det(D\phi^{i,n}_{t})(x)$. This implies that for any $A \in \mathcal{B}(O)$ we have

$$\left(\hat{\phi}^{i,e,n}_{t}\right)^* \tilde{\mu}^X(A) = \int_{A \times S} \det(D\hat{\phi}^{i,n,e}_{t})(x,s) \tilde{\mu}(dx,ds)$$

$$= \int_{A} \det(D\phi^{i,n}_{t})(x) \tilde{\mu}(dx,ds)$$

$$= \int_{A} \det(D\phi^{i,n}_{t})(x) \tilde{\mu}^X(dx).$$

Arguing as in the proof of Example 4.1 one obtains that there exists $k > 0$ such that $\tilde{\mu}^X = km$.

Now we look for the measure $\tilde{\mu}$. It follows from [Par67] that $\tilde{\mu}(dx,ds) = \tilde{\mu}_c(ds,x) \tilde{\mu}^X(dx)$ and we notice that for $t = 0$

$$\hat{\phi}^{i,n,g}_{0}(x,s) = (x,a(g,s)).$$

We proceed in order to show that $\tilde{\mu}_c(\cdot,x)$ is equivalent to $m_S$. To this end compute the following integral

$$\int_{O \times S} F(x,a(g,s)) \tilde{\mu}(dx,ds) = \int_{O \times S} F(x,s) \frac{d(g^* m_S)}{dm_S}(s) \tilde{\mu}(dx,ds)$$

$$= \int_{O \times S} F(x,s) \frac{d(g^* m_S)}{dm_S}(s) \tilde{\mu}_c(x,ds) \tilde{\mu}^X(dx).$$
But the left hand side integral is equal to
\[ \int_{O \times S} F(x, a(g, s)) \tilde{\mu}_c(x, ds) \tilde{\mu}^X(dx). \]
Since $G$ is locally compact, then by Proposition 4.11 for $\tilde{\mu}^X$-a.s. $x$ we have
\[ (g^* \tilde{\mu}_c)(x, ds) = \frac{d(g^* m_S)}{dm_S}(s) \tilde{\mu}_c(x, ds). \]
Then by [Mac51] we conclude that there exists a constant $k' > 0$ such that
\[ \tilde{\mu}_c(x, ds) = k'm_S(ds). \]
Putting all together we have
\[ \mu(dx, ds) = k'k m_S(ds)m_X(dx). \]

**Example 4.4** Let $X$ be a $d$-dimensional with volume element $m_X$. Let $S$ be a Banach manifold and a homogeneous space with respect to the topological group $G$ and a $G$ quasi-invariant measure $m_S$. Assume that there exists a family of locally-compact subgroups $(G_n)_{n \in \mathbb{N}}$ such that the group $G_0$ algebraically generated by $\cup_{n \in \mathbb{N}} G_n$ is dense in $G$. Furthermore, assume that the measure $m_S$ is characterized by its Radon-Nikodym derivative, i.e. if for each $n \in \mathbb{N}$ the measure $\tilde{m}$ is $G_n$ quasi-invariant with
\[ (g^* \tilde{m})(ds) = \frac{d(g^* m_S)}{dm_S}(s) \tilde{m}(ds) \]
then there exist a constant $k > 0$ such that $\tilde{m}_S = km_S$. Then $m = m_X \otimes m_S$ is a quasi-invariant measure with respect to $\text{Diff}_0(X \times S, G)$ constructed as in Example 4.3. The extra assumption on $m_S$ is needed to substitute the result of Mackey in the proof of Example 4.3 in order to obtain the analogous result.

Let $\mu$ be a $\text{Diff}_\text{small}(O \times S, \text{Diff}(S))$-quasi-invariant measure with Radon-Nikodym derivative of the form
\[ \frac{d((\hat{\phi}_{t,n,g}^i)^* \mu)}{d\mu}(x, s) = \frac{r((\hat{\phi}_{t,n,g}^i)^{-1}(x, s)) d((\hat{\phi}_{t,n,g}^i)^* m)}{r(x, s)}(x, s), \]
for a certain $r > 0$, $m$-a.s. Then there exists a constant $k > 0$ such that $\mu = km$.
4.2 Differential geometry on configuration space

We now recall the differentiable structure on \( \hat{\Gamma} \) which was introduced in [AKR98a] and its relations with \( K \)-transform see [KK02a] to where we refer for more details and historical remarks. On \( \hat{\Gamma}_0 \) we have a natural differentiable structure inherited from the manifold structure of the spaces \( \Gamma^{(n)}_\hat{X} \).

Since we have already introduced the \( K \)-transform which “lifts” functions on \( \hat{\Gamma}_0 \) to functions on \( \hat{\Gamma} \) (cf. Subsection 3.3), we will describe the differentiable structure in \( \hat{\Gamma} \) using the \( K \)-transform. Let \( \hat{v} \in \text{Vect}_0(\hat{X}) \) be a given vector field on \( \hat{X} \) and denote the corresponding flow by \( \phi^\hat{v}_t : \hat{X} \to \hat{X} \), \( t \in \mathbb{R} \) which is a one-parameter subgroup of diffeomorphisms.

Denote by \( \text{Exp}^\hat{v} \) the vector field on \( \hat{\Gamma}_0 \) which coincides on each \( \Gamma^{(n)}_\hat{X} \) with \( \sum_{i=1}^n \hat{v}(\hat{x}) \). The corresponding flow \( \phi^\text{Exp}^\hat{v}_t : \hat{\Gamma}_0 \to \hat{\Gamma}_0 \) is simply \( \phi^\text{Exp}^\hat{v}_t(\hat{\eta}) = \{ \phi^\hat{v}_t(\hat{x}) | \hat{x} \in \hat{\eta} \} \).

The directional derivative along \( \text{Exp}^\hat{v} \) is defined by

\[
\nabla^\hat{\Gamma}_0 \text{Exp}^\hat{v} G(\hat{\eta}) := \frac{d}{dt} G(\phi^\text{Exp}^\hat{v}_t(\hat{\eta}))|_{t=0},
\]

where \( G \in BC_{bs}^{1}(\hat{\Gamma}_0) \) (the set of all differentiable bounded functions with bounded support).

For all \( G \in BC_{bs}^{1}(\hat{\Gamma}_0) \) define the directional derivative by

\[
\nabla^\hat{\Gamma}_v KG(\hat{\gamma}) = (K \nabla^\hat{\Gamma}_0 \text{Exp}^\hat{v}) G(\hat{\gamma}) =, \quad \hat{\gamma} \in \hat{\Gamma}.
\]

These definitions coincides with the one given in [AKR98a], see also [KK02a, Lemma 7.1], i.e., the flow \( \phi^\hat{v} \) is lifted from \( \hat{X} \) to \( \hat{\Gamma} \) by

\[
\phi^\hat{v}_t : \hat{\Gamma} \to \hat{\Gamma}, \quad \hat{\gamma} \mapsto \phi^\hat{v}_t(\hat{\gamma}) = \{ \phi^\hat{v}_t(\hat{x}) | \hat{x} \in \hat{\gamma} \}
\]

and the directional derivative along this vector field \( \hat{v} \) is defined on functions \( F : \hat{\Gamma} \to \mathbb{R} \) by

\[
\nabla^\hat{\Gamma}_v F(\hat{\gamma}) := \frac{d}{dt} F(\phi^\hat{v}_t(\hat{\gamma}))|_{t=0}
\]

provided that the right hand side of (4.36) exists.

The tangent space \( T_{\hat{\gamma}}\hat{\Gamma} \) to the configuration space \( \hat{\Gamma} \) at \( \hat{\gamma} \in \hat{\Gamma} \) is defined as the set of all \( (\hat{v}(\hat{x}))_{\hat{x} \in \hat{\gamma}} \) with \( \hat{v}(\hat{x}) \in T_{\hat{x}}\hat{X} \) for all \( \hat{x} \in \hat{\gamma} \), and \( \sum_{\hat{x} \in \hat{\gamma}} |\hat{v}(\hat{x})|^2_{T_{\hat{x}}\hat{X}} < \infty \). It is equipped with the following scalar product

\[
\langle (\hat{v}^1(\hat{x}))_{\hat{x} \in \hat{\gamma}}, (\hat{v}^2(\hat{x}))_{\hat{x} \in \hat{\gamma}} \rangle_{T_{\hat{\gamma}}\hat{\Gamma}} = \sum_{\hat{x} \in \hat{\gamma}} \langle \hat{v}^1(\hat{x}), \hat{v}^2(\hat{x}) \rangle_{T_{\hat{x}}\hat{X}}.
\]
The construction of $T^\hat{\gamma}_\Gamma$ is such that the gradient corresponding to the directional derivative satisfy
\[
\nabla^\hat{\gamma} F(\hat{\gamma}) = \langle \nabla^\hat{\gamma} F(\hat{\gamma}), \hat{v} \rangle_{T^\hat{\gamma} \Gamma},
\]
where $\nabla^\hat{\gamma} F(\hat{\gamma}) = (\nabla^\hat{\gamma} F(\hat{\gamma}, \hat{x}))_{\hat{x}} \in T^\hat{\gamma}_\Gamma$. Obviously we have the following relation between the gradients in $\hat{\Gamma}$ and $\hat{\Gamma}_0$. For the proof we refer to [KK02a, Proposition 7.2].

**Proposition 4.5** For all $G \in BC^1_{bs}(\hat{\Gamma}_0)$ the following equality holds for any $\hat{\gamma} \in \hat{\Gamma}$
\[
\langle \nabla^\hat{\gamma} KG(\hat{\gamma}, \hat{x}), \hat{x} \rangle_{\hat{x} \in \hat{\gamma}} = \sum_{\hat{\eta} \in \hat{\gamma}, \hat{x} \in \hat{\eta}} \nabla^\hat{\eta} G(\hat{\eta}, \hat{x})_{\hat{x} \in \hat{\gamma}}.
\]

Finally we introduce the divergence on $\hat{\Gamma}$, denoted by $\text{div}^\hat{\Gamma}$ which is defined for any $\hat{v} \in \text{Vect}_\Lambda(\hat{X})$, $\Lambda \in \mathcal{L}(\hat{X})$ by
\[
\text{div}^\hat{\Gamma} \text{Exp}^\hat{\gamma} := \langle \text{div}^X \hat{v}, \hat{\gamma} \rangle = \sum_{\hat{x} \in \hat{\gamma}} \text{div}^X \hat{v}(\hat{x}).
\]

### 4.3 Characterization via Radon-Nikodym derivative

For the convenience of the reader and for completeness of this work we give a proof for the quasi-invariance of the marked Poisson measures and derive the Radon-Nikodym derivative.

**Theorem 4.6** The marked Poisson measure $\pi_\sigma$ is $\text{Diff}_{\text{max}}(\hat{X})$-quasi-invariant and the Radon-Nikodym derivative is given for any $\hat{\phi} \in \text{Diff}_{\text{max}}(\hat{X})$ by
\[
\frac{d(\hat{\phi}^* \pi_\sigma)}{d\pi_\sigma}(\hat{\gamma}) = \prod_{\hat{x} \in \hat{\gamma}} \frac{\hat{\rho}(\hat{\phi}^{-1}(\hat{x}))}{\hat{\rho}(\hat{x})} J_{\hat{\phi}^{-1}}(\hat{x}),
\]
where $J_{\hat{\phi}^{-1}}$ is the Jacobian determinant of $\phi^{-1}$ with respect to $m$.

**Remark 4.7** The intensity measure $\sigma$ is $\text{Diff}_{\text{large}}(\hat{X})$-quasi-invariant because $\sigma$ is of the form $\sigma(dx, ds) = \hat{\rho}(x, s)\hat{m}(dx, ds)$ and $\hat{\rho} > 0 \hat{m}$-a.a. The Radon-Nikodym derivative is just
\[
\frac{d(\hat{\phi}^* \sigma)}{d\sigma}(\hat{x}) = \frac{\hat{\rho}(\hat{\phi}^{-1}(\hat{x}))}{\hat{\rho}(\hat{x})} J_{\hat{\phi}^{-1}}(\hat{x}),
\]

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where the quotient is ill defined on a set of \( \hat{m} \)-measure zero and therefore also \( \sigma \)-measure zero.

**Proof.** Let \( \hat{\phi} \in \text{Diff}_{\text{large}}(\hat{X}) \) and \( F \) a positive cylinder function on \( \hat{\Gamma} \) be given. By definition there exists \( \Lambda \in \mathcal{B}_c(X) \) such that \( \text{supp}\hat{\phi} \subset \Lambda \times S \) and \( F \) is \( \mathcal{B}_\Lambda(\hat{\Gamma}) \)-measurable. Since \( \pi_\sigma \) is a canonical marked Gibbs measure for the potential \( V = 0 \), according to Definition 3.2 we have

\[
\int_{\hat{\Gamma}} F(\hat{\phi}(\gamma)) \sigma_\Lambda(d\gamma) = \int_{\hat{\Gamma}} \sigma(\Lambda)^{-|\hat{\gamma}|}_\Lambda \int_{\Gamma(\hat{\gamma})} F(\hat{\phi}(\eta)) \sigma_{|\hat{\gamma}|}(d\eta) \sigma(d\gamma).
\]

Therefore, the inner integral on the right hand side of the above expression is equal to

\[
\int_{\Lambda} F(\hat{\phi}(\{\hat{x}\}_1^{\hat{\gamma}|\Lambda|})) \sigma(d\hat{x})\rvert_1^{\hat{\gamma}|\Lambda|} = \int_{\Lambda} F(\{\hat{x}\}_1^{\hat{\gamma}|\Lambda|}) \prod_{\hat{x} \in \{\hat{x}\}_1^{\hat{\gamma}|\Lambda|}} \frac{\hat{\rho}(\hat{\phi}^{-1}(\hat{x}))}{\hat{\rho}(\hat{x})} J\hat{\phi}^{-1}(\hat{x}) \sigma(d\hat{x})\rvert_1^{\hat{\gamma}|\Lambda|}.
\]

The result of the theorem follows from the support properties of \( \hat{\phi} \) and once more by Definition 3.2.

All \( \text{Diff}_{\text{large}}(\hat{X}) \)-quasi-invariant measures have this form since local translations form a subgroup. In the next two lemmas we consider a family \( (M_i)_{i=1}^n \) of manifolds equipped with the group of diffeomorphisms \( \text{Diff}_{\text{small}}(M_i) \). It is assumed that these subgroups characterize measures by their Radon-Nikodym derivative. From this family we construct a new manifold \( M \) and embed \( \text{Diff}_{\text{small}}(M_i) \) as subgroups of \( \text{Diff}(M) \). We prove that they still characterize measures.

**Lemma 4.8** Let \( M_1, \ldots, M_n \) be orientable manifolds, \( r \) a measurable function on \( M := M_1 \times \cdots \times M_n \) such that \( r \geq 0 \) and \( \mu \) a measure on \( M \). For each \( i = 1, \ldots, n \) consider the subgroup \( \text{Diff}_{\text{small}}(M_i) \) of \( \text{Diff}_0(M_i) \) which characterizes quasi-invariant measures, see Subsection 4.1. If \( \mu \) is quasi-invariant with respect to \( \times_{i=1}^n \text{Diff}_{\text{small}}(M_i) \) with Radon-Nikodym derivative

\[
\frac{d(\phi^*\mu)}{d\mu}(x) = \mathbb{1}_{\{r(x) \neq 0\}}(x) \frac{r(\phi^{-1}(x))}{r(x)} J\phi^{-1}(x),
\]

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where $J\phi^{-1}$ is the Jacobin determinant w.r.t. $m$ (volume form on $M$), then
\[ \mu(dx) = kr(x)m(dx), \]
where $k$ is a constant.

**Remark 4.9** To say that a measure $\mu$ has a Radon-Nikodym derivative

\[ \frac{d\phi^*\mu}{d\mu}(x) = \mathbb{1}_{\{r \neq 0\}}(x) \frac{r(\phi^{-1}(x))}{r(x)} J\phi^{-1}(x), \quad r \geq 0, \]

is equivalent to assume that $r > 0$ $\mu$-a.s. and $\mu$ has Radon-Nikodym derivative

\[ \frac{d\phi^*\mu}{d\mu}(x) = \frac{r(\phi^{-1}(x))}{r(x)} J\phi^{-1}(x). \]

Indeed, because

\[
\int_M \mathbb{1}_{\{r=0\}}(x) \mu(dx) = \int_M \mathbb{1}_{\{r \circ \phi^{-1}=0\}}(\phi(x)) \mu(dx)
\]
\[
= \int_M \mathbb{1}_{\{r \circ \phi^{-1}=0\}}(x) \mathbb{1}_{\{r \neq 0\}}(x) \frac{r(\phi^{-1}(x))}{r(x)} J\phi^{-1}(x) \mu(dx)
\]
\[
= 0
\]

which implies that $\mu(\{r = 0\}) = 0$. According to quasi-invariance also $\mu(\{r \circ \phi = 0\}) = 0$ for all $\phi$. Practically the second assumption is more suitable for dealing. As it simplify the notation we use from time to time the first one.

**Proof.** Define a new measure $\tilde{\mu}$ on $M$ by $\tilde{\mu}(dx) := \mathbb{1}_{\{r \neq 0\}}(x) \frac{1}{r(x)} \mu(dx)$. Then for any positive measurable function $f$ on $M$ we have

\[
\int_M f(\phi(x)) \tilde{\mu}(dx) = \int_M f(\phi(x)) \mathbb{1}_{\{r \neq 0\}}(x) \frac{1}{r(x)} \mu(dx)
\]
\[
= \int_M f(x) \mathbb{1}_{\{r \circ \phi^{-1} \neq 0, r \neq 0\}}(x) \frac{r(\phi^{-1}(x))}{r(\phi^{-1}(x)) r(x)} J\phi^{-1}(x) \mu(dx)
\]
\[
= \int_M f(x) \mathbb{1}_{\{r \circ \phi^{-1} \neq 0\}}(x) J\phi^{-1}(x) \tilde{\mu}(dx)
\]
\[
= \int_M f(x) J\phi^{-1}(x) \mu(dx).
\]

The last equality holds because $\tilde{\mu}$ is absolutely continuous with respect to $\mu$ and $\mu(\{r \circ \phi^{-1} = 0\}) = 0$. We reduced the problem to the case $r = 1$. 29
The result will follows by induction on $n$. For $n = 1$ the result is true by definition since $\text{Diff}_{\text{small}}(X)$ characterizes $\mu$. Suppose now that for any $k = 1, \ldots, n - 1$ the result is valid, then we should prove that the result also holds for $k = n$. To this end let $f$ be a function on $M$ of the following form 

$f(x_1, \ldots, x_n) = h(x_n)g(x_1, \ldots, x_{n-1})$, $h, g$ continuous and compact support

and $\phi(x_1, \ldots, x_n) = (x_1, \ldots, x_{n-1}, \phi_n(x_n))$, where $\phi_n \in \text{Diff}_{\text{small}}(M_n)$. Then, by hypothesis we have

$$\int_{M_n} h(\phi_n(x_n)) \int_{M_1 \times \cdots \times M_{n-1}} g(x)^{n-1} \mu(dx)^n_1 = \int_{M_n} h(x_n)J\phi_1^{-1}(x_n) \int_{M_1 \times \cdots \times M_{n-1}} g(x)^{n-1}_1 \mu(dx)^n_1.$$

As $\text{Diff}_{\text{small}}(M_n)$ characterizes measures by Radon-Nikodym derivative there exits $k_g \in \mathbb{R}_+$ such that

$$\int_{M_n} h(x_n) \int_{M_1 \times \cdots \times M_{n-1}} g(x_1, \ldots, x_{n-1})\mu(dx)^n_1 = k_g \int_{M_n} h(x_n)m(dx_n). \quad (4.38)$$

Choose a function $f_0 \geq 0$ on $M_n$ such that

$$\int_{M_n} f_0(x_n)m(x_n) > 0,$$

then define $\tilde{\mu}$ on $M$ by

$$\int_{M_1 \times \cdots \times M_{n-1}} g(x)^{n-1}_1 d\tilde{\mu}(x)^{n-1}_1 := \int_{M_1 \times \cdots \times M_{n-1}} g(x)^{n-1}_1 \left(\int_{M_n} f_0(x_n)m(dx_n)\right)^{-1} \int_{M_n} f_0(x_n)\mu(dx)^n_1. \quad (4.39)$$

As a result we obtain $k_g = \int_{M_1 \times \cdots \times M_{n-1}} g(x_1, \ldots, x_{n-1})d\tilde{\mu}(x)^{n-1}_1$. Let us now prove that $\tilde{\mu}$ is $\times_{i=1}^{n-1} \text{Diff}_{\text{small}}(M_i)$-quasi-invariant. In fact, let $\tilde{\phi} \in \times_{i=1}^{n-1} \text{Diff}_{\text{small}}(M_i)$ be given. Then by the quasi-invariance of $\mu$ for the diffeomorphism $\tilde{\phi}(x_1, \ldots, x_n) = (\tilde{\phi}(x_1, \ldots, x_{n-1}), x_n) \in \text{Diff}_{\text{small}}(M_i)$ we have

$$\int_{M_1 \times \cdots \times M_{n-1}} g(\tilde{\phi}(x_1, \ldots, x_{n-1}))\tilde{\mu}(dx)^{n-1}_1$$

$$= \left(\int_{M_n} f_0(x_n)m(dx_n)\right)^{-1} \int_{M_1 \times \cdots \times M_{n-1}} g(\tilde{\phi}(x_1, \ldots, x_{n-1}))f_0(x_n)\mu(dx)^n_1$$

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\[
\left( \int_{M_n} f_0(x_n) m(dx_n) \right)^{-1} \int_{M_1 \times \ldots \times M_n} g(x_1, \ldots, x_{n-1}) f_0(x_n) J \tilde{\phi}^{-1}(x_1^{n-1}) \mu(dx_1^n) = \int_{M_1 \times \ldots \times M_{n-1}} g(x_1, \ldots, x_{n-1}) J \tilde{\phi}(x_1^{n-1}) \tilde{\mu}(dx_1^{n-1}).
\]

The induction hypothesis yields that
\[
\tilde{\mu}(dx_1^{n-1}) = km(dx_1^{n-1}).
\]

Then according to (4.38) and (4.39)
\[
\int_{M_n} h(x_n) \int_{M_1 \times \ldots \times M_{n-1}} g(x_1^{n-1}) \mu(dx_1^{n-1}) = k \int_{M_1 \times \ldots \times M_{n-1}} g(x_1^{n-1}) m(dx_1^{n-1}) \int_{M_n} h(x_n) m(dx_n).
\]

The following lemma is not a trivial corollary of the previous one because the group of diffeomorphisms under consideration is much smaller than \( \times_{i=1}^n \text{Diff}_{\text{small}}(X_i) \). More precisely, we consider an embedding of \( \text{Diff}_0(\hat{X}) \) into \( \text{Diff}_0(\Gamma_n^X) \) in the following way
\[
\{x_1, \ldots, x_n\} \mapsto \{\phi(x_1), \ldots, \phi(x_n)\},
\]

instead of the full group \( \text{Diff}_0(\Gamma_n^X) \), which has elements of the following form
\[
\{x_1, \ldots, x_n\} \mapsto \{\phi_1(x_1, \ldots, x_n), \ldots, \phi_n(x_1, \ldots, x_n)\}
\]
or \( \times_{i=1}^n \text{Diff}_{\text{small}}(X) \), which elements of the form \( \{x_1, \ldots, x_n\} \mapsto \{\phi_1(x_1), \ldots, \phi_n(x_n)\} \).

In order to prove that already this smaller group characterizes quasi-invariant measures we essentially use the fact that particle configurations can not contain particles with coinciding position.

**Lemma 4.10** Let \( \mu \) be a given probability measure on \( (\Gamma_n^X, \mathcal{B}(\Gamma_n^X)), \Lambda \in \mathcal{L}(\hat{X}) \) connected and open, \( n \in \mathbb{N} \). If \( \mu \) is quasi-invariant with respect to \( \text{Diff}_{\text{small}}(\hat{X}) \) with Radon-Nikodym derivative given by

\[
d(\hat{\phi}^* \mu)(\hat{\eta}) = \hat{r}(\hat{\phi}^{-1}(\hat{\eta})) \prod_{\hat{x} \neq 0}(\hat{\eta}) \prod_{\hat{x} \in \hat{\eta}} J \hat{\phi}^{-1}(\hat{x}), \forall \hat{\eta} \in \Gamma_n^X,
\]

where \( \hat{r} : \Gamma_n^X \rightarrow \mathbb{R}^+ \) is a measurable mapping which is \( \mu \)-a.s. positive. Then there exists a constant \( k \) such that \( \mu(d\hat{\eta}) = k \hat{r}(\hat{\eta}) \lambda_{\hat{\eta}}|_{\Gamma_n^X}(d\hat{\eta}) \).
Proof. Recall that for any \( n \in \mathbb{N} \) and \( \Lambda \in \mathcal{L}(\hat{X}) \), \( \Gamma_{\Lambda}^{(n)} \) is a manifold (cf. Subsection 2.1) with charts given by \( (\hat{U}_1 \times \cdots \times \hat{U}_n, \hat{h}_1 \times \cdots \times \hat{h}_n) \) where \( (\hat{U}_i, \hat{h}_i) \) are charts in \( \hat{X} \) such that \( U_i \in \mathcal{O}_c(\hat{X}) \) \( \hat{U}_i \cap \hat{U}_j = \emptyset, i \neq j \), see (2.5) for more details. Denote by \( \hat{O}_i := \hat{h}_i(\hat{U}_i) \) their open image in \( \mathbb{R}^d \), where \( d = \dim(\hat{X}) \). From now on we work on the open set \( \hat{O} := \hat{O}_1 \times \ldots \times \hat{O}_n \subset \mathbb{R}^{nd} \) and keep the same name for the objects transported from \( \Gamma_{\Lambda}^{(n)} \) to \( \hat{O} \).

For any measurable positive function \( F \) on \( \hat{O} \) and for all diffeomorphisms \( \hat{\phi}_i \in \text{Diff}_{\text{small}}(\hat{O}_i) \) we define \( \hat{\phi} := \hat{\phi}_1 \circ \ldots \circ \hat{\phi}_n \in \text{Diff}_{\text{small}}(\hat{O}) \). Notice as the \( \hat{U}_i, i = 1, \ldots, n \) are pairwise disjoint for \( x_1, \ldots, x_n \in \hat{O}_1 \times \ldots \times \hat{O}_n \) we have

\[
(\hat{\phi}(\hat{x}_1), \ldots, \hat{\phi}(\hat{x}_n)) = (\hat{\phi}_1(\hat{x}_1), \ldots, \hat{\phi}_n(\hat{x}_n)).
\]

Therefore using (4.40) we obtain

\[
\int_{\hat{O}} F(\hat{\phi}_1(\hat{x}_1), \ldots, \hat{\phi}_n(\hat{x}_n)) \mu(d\hat{x})^n_1 = \int_{\hat{O}} F(\hat{x})^n_1 \hat{r}(\hat{x}_1, \ldots, \hat{x}_n) \prod_{i=1}^n J(\hat{\phi}_i^{-1}(\hat{x}))_{1} \mu(d\hat{x})^n_1.
\]

As a result \( \mu \) as a measure on \( \hat{O} \) is quasi-invariant with respect to \( \times_{i=1}^n \text{Diff}_{\text{small}}(\hat{O}_i) \).

Thus, according to Lemma 4.8 there exists a constant \( k \) such that \( \mu(d\hat{x}_1, \ldots, d\hat{x}_n) = k \hat{r}(\hat{x}_1, \ldots, \hat{x}_n) \hat{m}(d\hat{x}_1, \ldots, d\hat{x}_n) \) on \( \hat{O} \). As \( \Gamma_{\Lambda}^{(n)} \) is connected we obtain the claimed result. \hfill \( \blacksquare \)

Proposition 4.11 Let \( (X, \mathcal{B}) \) be a Polish space (or a standard Borel space) and \( G \) a locally compact group with Haar measure \( m_G \) acting on \( X \). Let \( \mu \) be a probability measure on \( X \) quasi-invariant with respect to the action of \( G \) on \( X \). Take a countably generated sub-\( \sigma \)-algebra \( \mathcal{B}_0 \) of \( \mathcal{B} \) such that \( gB = B \) for all \( g \in G \) and all \( B \in \mathcal{B}_0 \). Then \( \mu(\cdot | \mathcal{B}_0) \) is quasi-invariant with respect to the action of \( G \) in \( X \) with the same Radon-Nikodym derivative.

Proof. Let \( g \in G \) be fixed, \( B_0 \in \mathcal{B}_0 \) and \( B \in \mathcal{B} \) be given. Then it is easy to see that

\[
\int_{B_0} \mu(g^{-1}B | \mathcal{B}_0)(y) \mu(dy) = \int_{B_0} \mu(B | \mathcal{B}_0)(y) \frac{d(g^* \mu)}{d\mu}(y) \mu(dy).
\]
In fact we have
\[
\int_{B_0} \mu(g^{-1}B|B_0)(y)\mu(dy) = \int_X \mathbb{1}_{B_0}(y) \mathbb{E}\left( (g^{-1}B|B_0)(y) \right)\mu(dy)
\]
\[
= \int_X \mathbb{E}\left( \mathbb{1}_{B_0}(y) \mathbb{1}_{g^{-1}B}(\cdot|B_0)(y) \right)\mu(dy)
\]
\[
= \int_X \mathbb{1}_{B_0}(y) \mathbb{1}_B(gy)\mu(dy)
\]
\[
= \int_X \mathbb{1}_{B_0}(g^{-1}y) \mathbb{1}_B(y) \frac{d(g^*\mu)}{d\mu}(y)\mu(dy)
\]
\[
= \int_{B_0} \int_B \frac{d(g^*\mu)}{d\mu}(y)\mu(dx|B_0)(y)\mu(dy).
\]

Hence for \( g \in G \) and \( B \in \mathcal{B} \) it holds for \( \mu \)-a.s. \( y \)
\[
\mu(g^{-1}B|B_0)(y) = \int_B \frac{d(g^*\mu)}{d\mu}(y)\mu(dx|B_0)(y). \tag{4.41}
\]

We now consider the class \( \tilde{\mathcal{B}} \subseteq \mathcal{B} \) for which (4.41) holds. Then it is not hard to see that \( \tilde{\mathcal{B}} \) forms a Dynkin class, the details we give in Lemma 4.13.

Therefore we have for \( g \in G \) that \( \mu \)-a.a. \( y \)
\[
(g^*\mu)(dx|B_0)(y) = \frac{d(g^*\mu)}{d\mu}(x)\mu(dx|B_0)(y).
\]

Let us define \( C \) by
\[
C := \left\{ (g, y) \in G \times X \mid (g^*\mu)(\cdot|B_0)(y) \neq \frac{d(g^*\mu)}{d\mu}(\cdot|B_0)(y) \right\}.
\]

We have \( C \in \mathcal{B}(G) \times \mathcal{B}(X) \),
\[
\int_X \mathbb{1}_C(g, y)\mu(dy) = 0,
\]
and thus
\[
\int_{G \times X} \mathbb{1}_C(g, y)\mu(dy)m_G(dg) = 0.
\]

Hence by Fubini’s theorem for \( \mu \)-a.a. \( y \) we have
\[
\int_G \mathbb{1}_C(g, y)m_G(dg) = 0. \tag{4.42}
\]
Let $y \in X$ for which (4.42) and define $G(y)$ by

$$
G(y) := \left\{ g \in G \mid \mu(g^{-1}B|B_0)(y) = \int_B \frac{d(g^*\mu)}{d\mu}(x)\mu(dx|B_0)(y) \right\}.
$$

(4.43)

We have $m_G(G(y)^c) = 0$ which implies that $G(y)$ has full $m_G$ measure. Since $G(y)$ is a group, analogy to a theorem of A. Weil we may deduce that $G = G(y)$. In other words, $\mu(\cdot|B_0)$ is quasi-invariant with respect to the action of $G$ with the same Radon-Nikodym derivative for $\mu$. $
$

**Corollary 4.12** If $G$ is a group algebraically generated by a countably family of local compact groups, then the result of Proposition 4.11 is also valid.

The following Lemma finish the proof of Proposition 4.11.

**Lemma 4.13**

1. Let $\tilde{B} \subset B$ be the class of sets for which (4.41) holds. Then $\tilde{B}$ forms a Dynkin system.

2. $G(y)$ defined by (4.43) is a group.

**Proof.**

1. We have to prove that

   (i) $X \in \tilde{B}$,

   (ii) if $B \in \tilde{B}$ then $B^c \in \tilde{B}$ and

   (iii) if $(B_n)_{n \in \mathbb{N}} \subset \tilde{B}$ is a pairwise disjoint sequence of sets in $\tilde{B}$, then $\bigcup_{n \in \mathbb{N}} B_n \in \tilde{B}$.

(i) To prove $X \in \tilde{B}$ we notice that

$$
\mu(g^{-1}X|B_0) = \int_X 1_X(gx)\mu(dx|B_0) = \mu(X|B_0) = 1.
$$

Hence it remains to show that

$$
\int_X \frac{d(g^*\mu)}{d\mu}(x)\mu(dx|B_0)(y) = 1.
$$

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If $B \in \mathcal{B}_0$, then
\[
\int_A \int_X \frac{d(g^* \mu)}{d\mu}(x) \mu(dx|\mathcal{B}_0)(y) \mu(dy) = \int_X \int_B \mathbb{1}_A(y) \frac{d(g^* \mu)}{d\mu}(x) \mu(dx|\mathcal{B}_0)(y) \mu(dy)
\]
\[
= \int_X \mathbb{1}_A(y) \frac{d(g^* \mu)}{d\mu}(y) \mu(dy)
\]
\[
= \int_X \mathbb{1}_A(y) \mu(dy)
\]
\[
= \mu(A)
\]

and from this follows (i).

(ii) Suppose now that $B \in \tilde{\mathcal{B}}$ in order to prove that $B^c \in \tilde{\mathcal{B}}$.

\[
\mu(g^{-1}B^c|\mathcal{B}_0) = \int_X \mathbb{1}_{B^c}(gx) \mu(dx|\mathcal{B}_0)
\]
\[
= (g^* \mu)(B^c|\mathcal{B}_0)
\]
\[
= 1 - (g^* \mu)(B|\mathcal{B}_0)
\]
\[
= 1 - \mu(g^{-1}B|\mathcal{B}_0)
\]
\[
= 1 - \int_B \frac{d(g^* \mu)}{d\mu}(x) \mu(dx|\mathcal{B}_0)
\]
\[
= \int_X \frac{d(g^* \mu)}{d\mu}(x) \mu(dx|\mathcal{B}_0) - \int_B \frac{d(g^* \mu)}{d\mu}(x) \mu(dx|\mathcal{B}_0)
\]
\[
= \int_{B^c} \frac{d(g^* \mu)}{d\mu}(x) \mu(dx|\mathcal{B}_0).
\]

(iii) Let $(B_n)_{n \in \mathbb{N}} \subset \tilde{\mathcal{B}}$ be as in the lemma. Then for each $n \in \mathbb{N}$ we have
\[
\mu(g^{-1}B_n|\mathcal{B}_0) = \int_{B_n} \frac{d(g^* \mu)}{d\mu}(x) \mu(dx|\mathcal{B}_0).
\]

Then the result follows by a limit argument.

2. Suppose that $g_1, g_2 \in G(y)$ in order to prove that $g_1g_2 \in G(y)$. On one hand
\[
\mu((g_1g_2)^{-1}B|\mathcal{B}_0) = \int_X \mathbb{1}_B(g_1g_2 x) \mu(dx|\mathcal{B}_0)
\]
\[
= \int_X \mathbb{1}_B(g_1 x)(g_2^* \mu)(dx|\mathcal{B}_0)
\]
\[= \int_X \mathbb{1}_B(x) \frac{d(g_2^* \mu)}{d \mu}(x) \mu(dx|\mathcal{B}_0)\]
\[= \int_X \mathbb{1}_B(x) \frac{d(g_2^* \mu)}{d \mu}(g_1 x) \frac{d(g_1^* \mu)}{d \mu}(x) \mu(dx|\mathcal{B}_0)\]
\[= \int_X \mathbb{1}_B(x) \frac{d((g_1 g_2)^* \mu)}{d \mu}(x) \mu(dx|\mathcal{B}_0).\]

The last equality follows from
\[((g_1 g_2)^* \mu)(A) = \int_A ((g_1 g_2)^* \mu)(dx)\]
\[= \int_A \frac{d(g_2^* \mu)}{d \mu}(x)(g_1^* \mu)(dx)\]
\[= \int_A \frac{d(g_2^* \mu)}{d \mu}(x) \frac{d(g_1^* \mu)}{d \mu}(x)(dx).\]

Hence \(g_1 g_2 \in G(y)\). It remains to prove that for each \(g \in G(y)\), then \(g^{-1} \in G(y)\). Assume for the moment that
\[
\left( \frac{d(g^* \mu)}{d \mu}(g x) \right)^{-1} = \frac{d((g^{-1})^* \mu)}{d \mu}(x). \tag{4.44}
\]

Then we have
\[\mu((g^{-1})^{-1} B|\mathcal{B}_0) = \int_X \mathbb{1}_B(x) \mu(dx|\mathcal{B}_0)\]
\[= \int_X \mathbb{1}_B(x) \left( \frac{d(g^* \mu)}{d \mu}(x) \right)^{-1} \frac{d(g^* \mu)}{d \mu}(x) \mu(dx|\mathcal{B}_0)\]
\[= \int_X \mathbb{1}_B(x) \left( \frac{d(g^* \mu)}{d \mu}(x) \right)^{-1} (g^* \mu)(dx|\mathcal{B}_0)\]
\[= \int_X \mathbb{1}_B(x) \frac{d((g^{-1})^* \mu)}{d \mu}(x) \mu(dx|\mathcal{B}_0)\]
\[= \int_X \frac{d((g^{-1})^* \mu)}{d \mu}(x) \mu(dx|\mathcal{B}_0).\]

Hence \(g^{-1} \in G(y)\). Finally we prove equality (4.44). But this follows with the same procedure as before applied to
\[\int_X \mathbb{1}_B(x) \left( \frac{d(g^* \mu)}{d \mu}(x) \right)^{-1} \frac{d((g^{-1})^* \mu)}{d \mu}(x) \mu(dx).\]
Now we are ready to characterize marked Poisson measures via their Radon-Nikodym derivative.

**Theorem 4.14** Let \( \mu \) be a probability measure on \( (\hat{\Gamma}, \mathcal{B}(\hat{\Gamma})) \) which is \( \text{Diff}_{\text{small}}(\hat{X}) \)-quasi-invariant with Radon-Nikodym derivative given, for any \( \hat{\phi} \in \text{Diff}_{\text{small}}(\hat{X}) \) by

\[
d(\hat{\phi}^* \mu)(\hat{\gamma}) = \prod_{\hat{x} \in \hat{\gamma}} \frac{\hat{\rho}^{-1}(\hat{x})}{\hat{\rho}(\hat{x})} \mathbb{1}_{\{\hat{\rho} \neq 0\}}(\hat{x}) J^{-1}(\hat{x}). \tag{4.45}
\]

Then, \( \mu \) is a canonical marked Gibbs measure for the potential \( V = 0 \), i.e., there exists a probability measure \( \zeta \) on \( \mathbb{R}_+^\gamma \) such that

\[
\mu = \int_0^\infty \pi_{\gamma} \zeta(dz), \quad \pi_0 := \delta_0. \tag{4.46}
\]

**Proof.** Let \( \Lambda \in \mathcal{L}(\hat{X}) \) be a given open and connected set and \( \hat{\phi} \) be an arbitrary element in \( \text{Diff}_{\text{small}}(\Lambda) \). Consider \( F \) of the form \( F = F_1 \cdot F_2 \) where \( F_1 \) is a positive \( \mathcal{B}_\Lambda(\hat{\Gamma}) \)-measurable function and \( F_2 \) a positive \( \mathcal{B}_{\Lambda^c}(\hat{\Gamma}) \)-measurable function. Then according to Lemma 3.4 we have

\[
\int_{\hat{\Gamma}} F(\hat{\phi}(\hat{\gamma})) d(\hat{\gamma}) = \int_{\hat{\Gamma}} F_2(\hat{\phi}(\hat{\gamma})) \mathbb{E}_\mu(F_1 \circ \hat{\phi}|\mathcal{F}_\Lambda)(\hat{\gamma}) d(\hat{\gamma}) \tag{4.47}
\]

\[
= \int_{\hat{\Gamma}} F_2(\hat{\phi}(\hat{\gamma})) \int_{\Gamma^{(\gamma,\Lambda)}} F_1(\hat{\phi}(\hat{\eta})) \mu_{\Lambda}(d\hat{\eta}, \hat{\gamma}) d(\hat{\gamma}) \tag{4.48}
\]

On the other hand using (4.45) we obtain

\[
\int_{\hat{\Gamma}} F(\hat{\phi}(\hat{\gamma})) d(\hat{\gamma}) = \int_{\hat{\Gamma}} F_1(\hat{\gamma}) F_2(\hat{\gamma}) \prod_{\hat{x} \in \hat{\gamma}} \frac{\hat{\rho}(\hat{x})}{\hat{\rho}^{-1}(\hat{x})} \mathbb{1}_{\{\hat{\rho} \neq 0\}}(\hat{x}) J^{-1}(\hat{x}) d(\hat{\gamma}) \]

\[
= \int_{\hat{\Gamma}} F_2(\hat{\gamma}) \mathbb{E}_\mu\left( F_1(\cdot) \prod_{\hat{x} \in \cdot} \frac{\hat{\rho}(\hat{x})}{\hat{\rho}^{-1}(\hat{x})} \mathbb{1}_{\{\hat{\rho} \neq 0\}}(\hat{x}) J^{-1}(\hat{x}) \middle| \mathcal{F}_\Lambda \right)(\hat{\gamma}) d(\hat{\gamma}) \]

\[
= \int_{\hat{\Gamma}} F_2(\hat{\gamma}) \int_{\Gamma^{(\gamma,\Lambda)}} F_1(\hat{\eta}) \prod_{\hat{x} \in \hat{\eta}} \frac{\hat{\rho}(\hat{x})}{\hat{\rho}^{-1}(\hat{x})} \mathbb{1}_{\{\hat{\rho} \neq 0\}}(\hat{x}) J^{-1}(\hat{x}) \mu_{\Lambda}(d\hat{\eta}, \hat{\gamma}) d(\hat{\gamma}).
\]
Since this result is valid for every $F_2$, we conclude that for $\mu$-a.a. $\hat{\gamma} \in \hat{\Gamma}$ we have

$$
\int_{\Gamma_{\Lambda}(\hat{\gamma})} F_1(\hat{\phi}(\hat{\eta})) \mu_\Lambda(d\hat{\eta}, \hat{\gamma}) = \int_{\Gamma_{\Lambda}(\hat{\gamma})} F_1(\hat{\eta}) \prod_{\hat{x} \in \hat{\eta}} \frac{\hat{\rho}(\hat{x})^{-1}(\hat{x})}{\hat{\rho}(\hat{x})} \mathbb{1}_{\{\hat{\rho} \neq 0\}}(\hat{x}) \hat{J}_{\hat{\gamma}}^{-1}(\hat{x}) \mu_\Lambda(d\hat{\eta}, \hat{\gamma}).
$$

Since $\text{Diff}_{\text{small}}(\hat{O})$ is countable for $\mu$-a.a. $\hat{\gamma} \in \hat{\Gamma}$ (4.49) holds simultaneously for all $\hat{\phi} \in \text{Diff}_{\text{small}}(\hat{O})$. Lemma 4.10 yields that

$$
\mu_\Lambda(d\hat{\eta}, \hat{\gamma}) = k \prod_{\{N_\Lambda = N_\Lambda(\hat{\gamma})\}} \hat{\rho}(\hat{x}) \hat{m}(d\hat{x}), \quad k > 0.
$$

Taking into account that $\mu_\Lambda(\Gamma_{\Lambda}, \hat{\gamma}) = 1 \hat{\gamma}$-\mu-a.s., then $k = \sigma(\Lambda)^{-N_\Lambda(\hat{\gamma})}$. Therefore $\mu_\Lambda$ is just the canonical specification for $V = 0$ and the statement of the theorem follows by (DLR)-equations. The representation (4.46) of $\mu$ is a consequence of the general theory of Gibbs measures, cf. [Pre76] or [Geo79].

### 4.4 Characterization via integration by parts

First of all we would like to compute the integration by parts formula for the mixed marked Poisson measures. They may be understood as canonical marked Gibbs measures for the potential $V = 0$.

**Theorem 4.15** Let $\mu$ be a probability measure on $(\hat{\Gamma}, \mathcal{B}(\hat{\Gamma}))$ of the following form

$$
\mu(d\hat{\gamma}) = \int_0^\infty \pi_{z\sigma}(d\hat{\gamma}) \zeta(dz), \quad \pi_0 := \delta_0,
$$

where $\zeta$ is a probability measure on $\mathbb{R}^+_0$ with the property

$$
\int_0^\infty z\zeta(dz) < \infty.
$$

Then, for all vector fields $\hat{v} \in \text{Vect}_{\text{max}}(\hat{X})$ and all $F \in \mathcal{F}C_{\text{b}}(\mathcal{D}, \hat{\Gamma})$ we have

$$
\int_{\hat{\Gamma}} \langle \nabla^\hat{f} F(\hat{\gamma}), \text{Exp}(\hat{v})(\hat{\gamma}) \rangle_{\hat{T}, \hat{f}} d\mu(d\hat{\gamma})
\quad = \quad -\int_{\hat{\Gamma}} F(\hat{\gamma}) \left[ \text{div}^\hat{f} \text{Exp}(\hat{v})(\hat{\gamma}) + \langle \text{Exp}(\hat{v})(\hat{\gamma}), \beta(\hat{\gamma}) \rangle_{\hat{T}, \hat{f}} \right] d\mu(d\hat{\gamma}),
$$

(4.52)
where the vector field \( \beta(\hat{\gamma}) \) is given by

\[
\beta(\hat{\gamma}) = (\beta_x(\hat{\gamma}))_{\hat{\gamma} \in \hat{\Gamma}} = \left( \frac{\nabla^X \hat{\rho}(\hat{x})}{\hat{\rho}(\hat{x})} 1_{\hat{\rho} \neq 0}(\hat{x}) \right)_{\hat{x} \in \hat{\gamma}} \in T_{\hat{\gamma}} \hat{\Gamma}.
\]

**Proof.** It is easy to compute the correlation measure corresponding to \( \mu \): let \( G \) be a positive measurable function on \( \hat{\Gamma}_0 \) be given, then we have

\[
\int_{\hat{\Gamma}} (KG)(\hat{\gamma}) \mu(d\hat{\gamma}) = \int_0^\infty \int_{\hat{\Gamma}} (KG)(\hat{\gamma}) \pi_{z\sigma}(d\hat{\gamma}) \zeta(dz) = \int_{\hat{\Gamma}_0} G(\hat{\eta}) \int_0^\infty \int_{\hat{X}_n} G(\{\hat{x}\}_1^n) (\sigma)_{\otimes n} (d\hat{x})_n \zeta(dz)
\]

This implies, according to (3.28), that

\[
\rho(\hat{\mu})(d\hat{\eta}) = \left( \int_0^\infty z^{[\hat{\eta}]} \zeta(dz) \right) \lambda_{\sigma}(d\hat{\eta}), \quad \forall \hat{\eta} \in \hat{\Gamma}_0,
\]

in particular

\[
\rho(\hat{\mu})(1)(d\hat{x}) = \left( \int_0^\infty x^{\zeta}(dz) \right) \sigma(d\hat{x}).
\]

Hence condition (4.51) is equivalent to finite first moment. We proceed the proof showing that all integrals in (4.52) are finite. To this end let us denote by \( C \) the upper bound of \( |F| \) and \( |\nabla^F F| \), i.e.,

\[
C := \sup_{\hat{\gamma} \in \hat{\Gamma}} \{ |F(\hat{\gamma})|, |\nabla^F F(\hat{\gamma})| \}.
\]

Using (4.54), we estimate the left hand side of (4.52) by

\[
\int_{\hat{\Gamma}} |\langle \nabla^F F(\hat{\gamma}), \text{Exp}(\hat{\nu}(\hat{\gamma})) \rangle_{T_{\hat{\gamma}} \hat{\Gamma}}| \mu(d\hat{\gamma}) \leq C \int_{\hat{X}} |\hat{\nu}(\hat{x})|_{T_{\hat{x}} \hat{\Gamma}} \sigma(d\hat{x}) \int_0^\infty z^{\zeta}(dz).
\]

According to the support of \( \hat{\nu} \) and the local finiteness of \( \sigma \) the right hand side of the above inequality is finite. The integral on the right hand side of
(4.52) we can estimate as follows
\[
\int_\Gamma \left| F(\hat{\gamma}) \right| \left| \text{div}^\mathcal{X} \text{Exp}(\hat{v})(\hat{\gamma}) + \langle \text{Exp}(\hat{v})(\hat{\gamma}), \beta(\hat{\gamma}) \rangle_{T_\gamma \Gamma} \right| \mu(d\hat{\gamma}) \tag{4.55}
\]
\[
\leq C \int_\mathcal{X} \left| \text{div}^\mathcal{X} \hat{v}(\hat{x}) \right| \sigma(d\hat{x}) + C \int_\mathcal{X} \left| \hat{v}(\hat{x}) \right|_{T_\hat{x} \mathcal{X}} \left| \nabla^\mathcal{X} \hat{\rho}(\hat{x}) \right|_{\hat{\rho}} \sigma(d\hat{x}).
\]
Since \( \hat{v} \in \text{Vect}_{\text{max}}(\mathcal{X}) \) and \( \hat{\rho} \in H^1_{\text{loc}}(\mathcal{X}) \), both integrals in the right hand side of (4.55) are finite. As \( \hat{v} \in \text{Vect}_{\text{max}}(\mathcal{X}) \) there exists \( \Lambda \in \mathcal{B}_\mathcal{X}(\mathcal{X}) \) such that \( \text{supp} \hat{\rho} \subset \Lambda \times S \). Let \( F \) be a \( \mathcal{B}_\Lambda(\Gamma) \)-measurable function. This allows us to write
\[
\int_\Gamma \langle \nabla^\mathcal{X} F(\hat{\gamma}), \text{Exp}(\hat{v})(\hat{\gamma}) \rangle_{T_\gamma \Gamma} \mu(d\hat{\gamma}) \tag{4.56}
\]
\[
= \int_\Gamma \sigma(\Lambda)^{[\gamma_\Lambda]} \int_{\Lambda^{[\gamma_\Lambda]}} \sum_{i=1}^{[\gamma_\Lambda]} \langle \nabla^\mathcal{X} \hat{v}(\hat{x}_i), \hat{v}(\hat{x}_i) \rangle_{T_{\hat{x}_i} \mathcal{X}} \sigma(d\hat{x})^{[\gamma_\Lambda]} \mu(d\hat{\gamma}).
\]
Here we used the fact that the measures \( \mu \) are just canonical marked Gibbs measures for the potential \( V = 0 \), cf. Definition 3.2. Apply the usual integration by parts formula on manifolds to the previous expression and one obtains
\[
- \int_\Gamma \sigma(\Lambda)^{[\gamma_\Lambda]} \int_{\Lambda^{[\gamma_\Lambda]}} \sum_{i=1}^{[\gamma_\Lambda]} \text{div}^\mathcal{X} \hat{v}(\hat{x}_i) \left[ \left. \left. \int_{\gamma_i}^{[\gamma_\Lambda]} \hat{v}(\hat{x}_i) \right|_{\nabla^\mathcal{X} \hat{\rho}(\hat{x}_i)} \mathbb{1}_{\{\hat{\rho} \neq 0\}} \right|_{T_{\hat{x}_i} \mathcal{X}} \right] \sigma(d\hat{x})^{[\gamma_\Lambda]} \mu(d\hat{\gamma}).
\]
Finally, we may rewrite this last result as
\[
- \int_\Gamma \int_{\Gamma^{[\gamma_\Lambda]}} \left. \left. F(\hat{\eta}) \text{div}^\mathcal{X} \text{Exp} \hat{\rho}(\hat{\eta}) + \left. \langle \text{Exp} \hat{\rho}(\hat{\eta}), \beta(\hat{\eta}) \rangle_{T_\hat{\eta} \Gamma^{[\gamma_\Lambda]}} \right|_{T_{\hat{\eta}} \Gamma^{[\gamma_\Lambda]}} \Pi_{\Lambda}(d\hat{\eta}, \hat{\gamma}) \right| \mu(d\hat{\gamma}),
\]
where the vector field \( \beta(\hat{\eta}) \in T_{\hat{\eta}} \Gamma^{[\gamma_\Lambda]} \) is given by
\[
\beta(\hat{\eta}) = \left. \left. \left( \nabla^\mathcal{X} \hat{\rho}(\hat{x}) \right|_{\hat{x}} \mathbb{1}_{\{\hat{\rho} \neq 0\}} \right|_{\hat{\eta}} \right) \in T_{\hat{\eta}} \Gamma^{[\gamma_\Lambda]}.
\]
The integration by parts formula (4.52) follows from (4.57) applying Definition 3.2. \( \blacksquare \)
Lemma 4.16 Let $M_1, \ldots, M_n$ be orientable manifolds, $r$ a measurable function on $M := M_1 \times \ldots \times M_n$ such that $r \geq 0$, $r \in H^{1,1}(M, m)$ and $\mu$ a measure on $M$. For each $i = 1, \ldots, n$ consider the set $\text{Vect}_{\text{small}}(M_i)$ which characterizes measures on $M_i$ by integration by parts. Let $\hat{v} \in \times_{i=1}^n \text{Vect}_{\text{small}}(M_i)$ be such that $\text{supp} \hat{v} \subset O$, $O \in \mathcal{O}_c(M)$ and $F \in C'^\infty(O)$, then if the following integration by parts is valid

$$\int_O \langle \nabla^\hat{X} F(\hat{x}), v(\hat{x}) \rangle d\hat{x} M \mu(d\hat{x}) = - \int_O F(\hat{x}) [\text{div}^\hat{X} \hat{v}(\hat{x}) + \langle \hat{v}(\hat{x}), \beta(\hat{x}) \rangle] d\hat{x},$$

for $\beta(\hat{x}) = \frac{\nabla^\mu r(\hat{x})}{r(\hat{x})}$, where $r \in H^{1,1}(M, m)$ and $r > 0$ $\mu$-a.s., then we have $\mu(dx) = kr(x)m(dx)$. Here $m$ is the Riemannian volume or some volume form on $M$ and $k \geq 0$ is a constant.

Proof. If we assume in addition that $r$ is continuous and positive, then the proof of this lemma is a straightforward adaptation of the proof of Lemma 4.8.

Lemma 4.17 Let $\mu$ be a probability measure on $(\Gamma^{(n)}_\Lambda, B(\Gamma^{(n)}_\Lambda))$, $n \in \mathbb{N}$ and $\Lambda \in \mathcal{L}(\hat{X})$ be given. If $\mu$ fulfills for all vector fields $\hat{v} \in \text{Vect}_{\text{small}}(\hat{X})$ the following integration by parts formula

$$\int_{\Gamma^{(n)}_\Lambda} \langle \nabla^\hat{X} F(\hat{\eta}), \text{Exp}(\hat{v})(\hat{\eta}) \rangle_{T_{\hat{\eta}}\Gamma^{(n)}_\Lambda} \mu(d\hat{\eta})$$

$$= - \int_{\Gamma^{(n)}_\Lambda} F(\hat{\eta}) [\text{div}^\hat{X} \text{Exp}(\hat{v})(\hat{\eta}) + \langle \text{Exp}(\hat{v})(\hat{\eta}), \beta(\hat{\eta}) \rangle]_{T_{\hat{\eta}}\Gamma^{(n)}_\Lambda} \mu(d\hat{\eta}), \quad (4.58)$$

for all $F \in C^1_0(\Gamma^{(n)}_\Lambda)$ with $\beta(\hat{\eta}) = \frac{\nabla^r \mu (\hat{\eta})}{r(\hat{\eta})}$, where $r \in H^{1,1}(\Gamma^{(n)}_\Lambda, \lambda_m)$, then $\mu(d\hat{\eta}) = kr(\hat{\eta})\lambda_m|\Gamma^{(n)}_\Lambda|$, $k$ constant.

Proof. Let $(\hat{U}_i, \hat{h}_i)_{i=1}^n$ and $\hat{O}$ be as in the proof of Lemma 4.10 and take $\hat{v}_i \in \text{Vect}_{\text{small}}(\hat{O}_i)$. We keep the same convention for notation as in Lemma 4.10. The above integration by parts formula gives, for the vector field $\hat{v} := \sum_{i=1}^n \hat{v}_i$ and $F \in C^1_0(\Gamma^{(n)}_\Lambda)$

$$\int_{\hat{O}} \sum_{i=1}^n \langle \nabla^\hat{X} F(\hat{x})^n, \hat{v}_i(\hat{x}_i) \rangle_{T_{\hat{x}_i}\hat{X}} \mu(d\hat{x})_1^n = - \int_{\hat{O}} F(\hat{x})_1^n \sum_{i=1}^n \text{div}^\hat{X} \hat{v}_i(\hat{x}_i) \mu(d\hat{x})_1^n$$

$$\quad - \int_{\hat{O}} F(\hat{x})_1^n \sum_{i=1}^n \langle \hat{v}_i(\hat{x}_i), \nabla^\hat{X} \sqrt{r(\hat{x})_1^n} \rangle_{T_{\hat{x}_i}\hat{X}} \mu(d\hat{x})_1^n \quad (4.59)$$
Notice that \( \hat{v}(\hat{x}_i) = \hat{v}_i(\hat{x}_i) \) due to the fact that the \( \hat{U}_i \), \( i = 1, \ldots, n \) are pairwise disjoint. Equation (4.59) is nothing than integration by parts formula on \( \hat{O} \) for all vectors fields \( (\hat{v}_1 \times \ldots \times \hat{v}_n) \in \times_{i=1}^n \text{Vect}_{\text{small}}(\hat{O}_i) \) and, therefore, according to Lemma 4.16 we have
\[
\mu(d\hat{x}_1, \ldots, d\hat{x}_n) = k\hat{r}(\hat{x}_1, \ldots, \hat{x}_n)\hat{m}(d\hat{x}_1, \ldots, d\hat{x}_n).
\]
This has to be shown.

**Theorem 4.18** Let \( \mu \) be a probability measure on \((\hat{\Gamma}, \mathcal{B}(\hat{\Gamma}))\) such that the first correlation measure \( \rho^{(1)}_\mu \) is absolutely continuous with respect to \( \sigma \) and there exists a constant \( C_R \) such that
\[
0 \leq \frac{d\rho^{(1)}_\mu}{d\sigma} \leq C_R.
\]
Assume that for all \( \hat{v} \in \text{Vect}_{\text{small}}(\hat{X}) \) we have the following integration by parts formula for any cylinder function \( F \in \mathcal{F}C_\infty^b(\hat{D}, \hat{\Gamma}) \)
\[
\begin{align*}
\int_{\hat{\Gamma}} \langle \nabla F(\hat{\gamma}), \text{Exp}(\hat{v})(\hat{\gamma}) \rangle_{T_{\hat{\gamma}}\hat{\Gamma}} \mu(d\hat{\gamma}) \\
= - \int_{\hat{\Gamma}} F(\hat{\gamma}) \left[ \text{div}^F \text{Exp}(\hat{v})(\hat{\gamma}) + \langle \text{Exp}(\hat{v})(\hat{\gamma}), \beta(\hat{\gamma}) \rangle_{T_{\hat{\gamma}}\hat{\Gamma}} \right] \mu(d\hat{\gamma}), \quad (4.60)
\end{align*}
\]
where \( \beta(\hat{\gamma}) \) is the vector field defined in (4.53). Then \( \mu \) is a canonical Gibbs measure for the potential \( V = 0 \), i.e., there exists a probability measure \( \zeta \) on \( \mathbb{R}_+^\infty \) such that
\[
\mu = \int_0^\infty \pi_\sigma \zeta(d\pi), \quad \pi_0 := \delta_0.
\]

**Proof.** By our assumption of the first correlation measure the integrals exists absolutely. Let \( \Lambda \in \mathcal{L}(\hat{X}) \) be open and \( \hat{v} \in \text{Vect}_{\text{small}}(\Lambda) \) a given vector field. Consider \( F_1, F_2 \in \mathcal{F}C_\infty^b(\hat{D}, \hat{\Gamma}) \) where \( F_1 \) is \( \mathcal{B}_\Lambda(\hat{\Gamma}) \)-measurable and \( F_2 \) is \( \mathcal{B}_\Lambda(\hat{\Gamma}) \)-measurable cylinder functions. Then the left hand side of (4.60) may be written using \( F = F_1F_2 \) as
\[
\int_{\hat{\Gamma}} \langle \nabla F(\hat{\gamma}), \text{Exp}(\hat{v})(\hat{\gamma}) \rangle_{T_{\hat{\gamma}}\hat{\Gamma}} \mu(d\hat{\gamma}) = \int_{\hat{\Gamma}} F_2(\hat{\gamma}) \mathbb{E}_\mu((\nabla F_1, \text{Exp}(\hat{v}))_{TT}|\mathcal{F}_\Lambda)(\hat{\gamma}) \mu(d\hat{\gamma}).
\]
Using the regular conditional expectation of \( \mu \) we obtain
\[
= \int_{\hat{\Gamma}} F_2(\hat{\gamma}) \int_{\mathcal{L}_\Lambda(\hat{\gamma} \Lambda)} \langle \nabla F_1(\hat{\eta}), \text{Exp}(\hat{v})(\hat{\eta}) \rangle_{T_{\hat{\eta}}\hat{\Gamma}(\hat{\gamma} \Lambda)} \mu_\Lambda(\hat{\eta}, \gamma) \mu(d\hat{\gamma}). \quad (4.61)
\]
On the other hand, the right hand side of (4.60) gives

$$- \int_{\hat{\Gamma}} F_2(\hat{\gamma}) \int_{\Gamma^{(\Lambda\hat{\gamma})}} F_1(\hat{\eta}) \left( \text{div} \exp(\hat{v})(\hat{\eta}) + \langle \exp(\hat{v})(\hat{\eta}), \beta(\hat{\eta}) \rangle_{T_{\hat{\eta}}\Gamma_{\Lambda}(\hat{\gamma})} \right) \mu_{\Lambda}(d\hat{\eta}, \hat{\gamma}) \mu(d\hat{\gamma}).$$

(4.62)

Since (4.61) and (4.62) are equal for any $F_2$ and $\text{Vect}_{\text{small}}(\hat{O})$ is countable it follows that for $\mu$-a.a. $\hat{\gamma}$ we have

$$\int_{\Gamma^{(\Lambda\hat{\gamma})}} \left( \text{div} F_1(\hat{\eta}) \exp(\hat{v})(\hat{\eta}) \right)_{T_{\hat{\eta}}\Gamma_{\Lambda}(\hat{\gamma})} \mu_{\Lambda}(d\hat{\eta}, \hat{\gamma})$$

$$= \int_{\Gamma^{(\Lambda\hat{\gamma})}} F_1(\hat{\eta}) \left( \text{div} \exp(\hat{v})(\hat{\eta}) + \langle \exp(\hat{v})(\hat{\eta}), \beta(\hat{\eta}) \rangle_{T_{\hat{\eta}}\Gamma_{\Lambda}(\hat{\gamma})} \right) \mu_{\Lambda}(d\hat{\eta}, \hat{\gamma}),$$

According to Lemma 4.17 we may conclude that

$$\mu_{\Lambda}(d\hat{\eta}, \hat{\gamma}) = k \mathbb{I}_{\{N_{\Lambda}(\hat{\gamma})\}}(\hat{\eta}) \prod_{\hat{x} \in \hat{\eta}} \hat{\rho}(\hat{x}) \hat{m}(d\hat{x}).$$

Taking into account that $\mu_{\Lambda}(\Gamma_{\Lambda}, \hat{\gamma}) = 1$ for $\mu$-a.a. $\hat{\gamma}$, then $k = \sigma(\Lambda)^{-N_{\Lambda}(\hat{\gamma})}$. This implies that $\mu_{\Lambda}$ is the canonical specification for $V = 0$ and the statement of the theorem follows by (DLR)-equations. The representation of $\mu$ is a consequence of the general theory of Gibbs measures, cf. [Pre76] or [Geo79].
5 Characterization of canonical marked Gibbs measures

This section is devoted to the analysis of canonical marked Gibbs measures. Namely, quasi-invariance properties and integration by parts formula. In Subsection 5.2 we give a necessary and sufficient condition in order that the Radon-Nikodym derivative of a probability measure $\mu$ on the configuration space $\hat{\Gamma}$ be a canonical marked Gibbs measure. In Subsection 5.3 we obtain analogous results for integration by parts. Finally, it remains to add that Subsection 5.1 consists of the necessary conditions on the interaction $V$ under which we obtain our results.

5.1 Conditions on the interactions

All assumptions concerning the potential $V$ are collected in this subsection. If $\mu$ is a tempered measure in the sense of Ruelle, cf. [Rue70] and the potential $V$ is ($S$), lower and upper regular, and $\nabla V$ is only upper and lower regular one sees easily that Assumption 3 are fulfilled, cf. Remark 5.2. First we give a general proposition of which all the remaining results are essentially corollaries, cf. Corollary 5.4 below. In this section we assume that all measures $\mu$ on $(\hat{\Gamma}, \mathcal{B}(\hat{\Gamma}))$ are from $\mathcal{M}_{1}^{ fm}(\hat{\Gamma})$.

Proposition 5.1 Let $\mu \in \mathcal{M}_{1}^{ fm}(\hat{\Gamma})$ and $G : \hat{\Gamma} \rightarrow \mathbb{R}^{+} \cup \{\infty\}$ be a given $\mathcal{B}(\hat{\Gamma})$-measurable function. If

$$\int_{\hat{\Gamma}} (G(\hat{x}) \wedge 1) \rho^{(1)}_{\mu}(d\hat{x}) < \infty \quad (5.63)$$

and

$$\rho^{(1)}_{\mu} \left( \left\{ \hat{x} \in \hat{\Gamma} \mid G(\hat{x}) = \infty \right\} \right) = 0,$$

then the series $\sum_{\hat{x} \in \hat{\Gamma}} G(\hat{x})$ is $\mu$-a.s. convergent.

Proof. Denote by $A$ the following subset subset of $\hat{\Gamma}$

$$A := \{ \hat{x} \in \hat{\Gamma} \mid F(\hat{x}) \leq 1 \}.$$ 

Taking into account that $(K \mathbb{1}_{A}G)(\hat{\gamma}) = \sum_{\hat{x} \in \gamma_{A}} G(\hat{x})$, we obtain that

$$\mathbb{E}_{\mu} \left( \sum_{\hat{x} \in \gamma_{A}} G(\hat{x}) \right) = \int_{\hat{\Gamma}} (K \mathbb{1}_{A}G)(\hat{\gamma}) \mu(d\hat{\gamma})$$
\[
\int_A G(\hat{x})\rho^{(1)}_\mu(d\hat{x}) \\
\leq \int_{\hat{X}} (G(\hat{x}) \wedge 1)\rho^{(1)}_\mu(d\hat{x}) < \infty.
\]

This implies that \( \sum_{\hat{x} \in \hat{\gamma}_A} G(\hat{x}) \) is \( \mu \)-a.s. absolutely convergent. On the other hand, the sum \( \sum_{\hat{x} \in \hat{\gamma}_{\Lambda^c}} |G(\hat{x})| \) contains only finite many summands. Indeed, because of (5.63)

\[
\mathbb{E}_\mu(N_{\Lambda^c}(\cdot)) = \int_{\Lambda^c} \rho^{(1)}_\mu(d\hat{x}) \leq \int_{\hat{X}} (G(\hat{x}) \wedge 1)\rho^{(1)}_\mu(d\hat{x}) < \infty.
\]

Furthermore, \( \mu \)-a.s. all these summands are finite

\[
\mu \left( \left\{ \hat{\gamma} \in \hat{\Gamma} \mid \text{exists } \hat{x} \in \hat{X} \text{ s.t. } G(\hat{x}) = \infty \right\} \right) \leq \int_{\hat{X}} \sum_{\hat{x} \in \hat{\gamma}} \mathbb{1}_{G^{-1}(\{\infty\})}(\hat{x}) \mu(d\hat{x})
\]

\[
= \rho^{(1)}_\mu \left( \left\{ \hat{x} \in \hat{X} \mid G(\hat{x}) = \infty \right\} \right)
\]

\[
= 0.
\]

Together this implies that \( \sum_{\hat{x} \in \hat{\gamma}_{\Lambda^c}} |F(\hat{x})| < \infty. \]

\[\boxed{}\]

**Assumption 1 (Stability)** There exists a \( B \geq 0 \) such that for all \( \Lambda \in \mathcal{O}_e(\hat{X}) \) and all \( \hat{\gamma} \in \Gamma_\Lambda \) we have

\[
E_\Lambda(\hat{\gamma}) \geq -B|\hat{\gamma}|.
\]

**Assumption 2 (No hard core)** For each \( \delta > 0 \) we have

\[
\sup_{(x,s),(y,t) \in \hat{X} \atop d(x,y) > \delta} V((x,s),(y,t)) < \infty.
\]

**Assumption 3 (Regularity)** For all \( \Lambda \in \mathcal{L}(\hat{X}) \) we have

\[
\int_{\hat{X}} (\sup_{\hat{x} \in \Lambda} |V(\hat{x}, \hat{y})| \wedge 1) \sigma(d\hat{y}) < \infty.
\]

**Remark 5.2** If \( \hat{X} \) is a vector space then it is sufficient to assume that \( V \) is upper and lower regular to guarante Assumption 3, see [Rue70]. In explicit, there exists a monotonic decreasing integrable function \( \psi \) and \( R > 0 \) such that \( |V(\hat{x}, \hat{y})| \leq \psi(|x-y|) \) for all \( \hat{x}, \hat{y} \) with \( |x-y| \geq R \) and \( \int_0^\infty |\psi(r)|r^{d-1}dr < \infty. \)
**Assumption 4** Let $\mu \in M^1_{fm}(\hat{\Gamma})$ be given such that the first correlation measure $\rho^{(1)}_{\mu}$ corresponding to $\mu$ is absolutely continuous with respect to $\sigma$ and we have
\[
\frac{d\rho^{(1)}_{\mu}}{d\sigma}(\hat{x}) \leq C_1, \quad \text{for some } C_1 > 0.
\]

**Remark 5.3** If $\mu$ is a measure on $\Gamma_{\mathbb{R}^d}$ then all assumptions are fulfilled if the conditions of [Rue70] and Assumption 2 holds, i.e. $\mu$ is tempered in the sense of D. Ruelle and there exist a $R > 0$, and positive bounded decreasing functions $\psi_1 : (0, R] \to \mathbb{R}$, $\psi_2 : [R, \infty) \to \mathbb{R}$ with $\int_0^R \psi_1(r)r^{d-1}dr = \infty$ and $\int_R^\infty \psi_2(r)r^{d-1}dr < \infty$ such that $V(x, y) \geq \psi_1(|x - y|)$ for $|x - y| \leq R$ and $|V(x, y)| \leq \psi_2(|x - y|)$ for $|x - y| \geq R$. In particular, $V$ is then superstable and Assumption 4 is not anymore necessary to obtain Corollary 5.4. Assumption 4 might be replaced by a support condition for $\mu$. For hard core potentials the Gibbs measure is not even quasi-invariant w.r.t. $\text{Diff}_{\text{small}}(X)$. Essential supremum w.r.t. $\sigma_2$ would be sufficient in Assumption 2, however $V$ is typically continuous for $x \neq y$.

**Corollary 5.4** Let $\mu \in M^1_{fm}(\hat{\Gamma})$ be a measure which fulfills Assumption 4, and $V$ a potential satisfying Assumptions 1-3. Let $\Lambda \in \mathcal{O}_c(\hat{X})$ and $\sigma(\partial \Lambda) = 0$. Then for $\mu$-a.a. $\hat{\gamma} \in \hat{\Gamma}$
\[
\sum_{\hat{x} \in \hat{\gamma}_{\Lambda^c}} \sup_{\hat{y} \in \Lambda} |V(\hat{x}, \hat{y})| < \infty,
\]
and $0 < Z_\Lambda(\hat{\gamma}) \leq \Xi_\Lambda(\hat{\gamma}) < \infty$. Moreover, for all $\hat{x} \in \Lambda$ the sum
\[
W(\{\hat{x}\}, \gamma_{\Lambda^c}) = \sum_{\hat{y} \in \hat{\gamma}_{\Lambda^c}} V(\hat{x}, \hat{y})
\]
is absolutely convergent.

**Proof.** Apply Proposition 5.1 for $G(\hat{y}) := 1_{\Lambda^c}(\hat{y}) \sup_{\hat{x} \in \Lambda} |V(\hat{x}, \hat{y})|$. Note that
\[
\int_{\hat{X}} (G(\hat{y}) \wedge 1)\rho^{(1)}_{\mu}(d\hat{y}) \leq C_1 \int_{\hat{X}} (\sup_{\hat{x} \in \Lambda} |V(\hat{x}, \hat{y})| \wedge 1)\sigma(d\hat{y})
\]
and $G(\hat{y}) < \infty$ if $\hat{y} \notin \partial \Lambda$. Hence, for $\mu$-a.a. $\hat{\gamma}$ there exists a constant $C_\Lambda(\hat{\gamma})$ such that
\[
\sum_{\hat{x} \in \hat{\gamma}_{\Lambda^c}} \sup_{\hat{y} \in \Lambda} |V(\hat{x}, \hat{y})| \leq C_\Lambda(\hat{\gamma}).
\]
Since always $Z_\Lambda(\hat{\gamma}) \leq \Xi_\Lambda(\hat{\gamma})$, it remains to show that $\Xi_\Lambda(\hat{\gamma}) < \infty$ and $Z_\Lambda(\hat{\gamma}) > 0 \mu$-a.s. Let $\hat{\eta} \in \Gamma_\Lambda$ be given, then we have
\[ W(\hat{\eta}, \hat{\gamma}_\Lambda) \leq C_\Lambda(\hat{\gamma})|\hat{\eta}| \]
and therefore
\[ \Xi_\Lambda(\hat{\gamma}) = \int_{\Gamma_\Lambda} e^{-E_\Lambda(\hat{\eta}) - W(\hat{\eta}, \gamma_\Lambda^e)} \pi_\sigma(d\hat{\eta}) \leq \int_{\Gamma_\Lambda} e^{B|\hat{\eta}| + C_\Lambda(\hat{\gamma})|\hat{\eta}|} \pi_\sigma(d\hat{\eta}) \leq \exp \left( \sigma(\Lambda)(e^{B + C_\Lambda(\hat{\gamma})} - 1) \right) < \infty. \]

On the other hand to prove that $Z_\Lambda(\hat{\gamma}) > 0$ we proceed as follows
\[ Z_\Lambda(\hat{\gamma}) = \int_{\Gamma_\Lambda^{(\hat{\gamma}_\Lambda)}} e^{-E(\hat{\eta}) - W(\hat{\eta}, \gamma_\Lambda^e)} \pi_\sigma(d\hat{\eta}) \geq \int_{\Gamma_\Lambda^{(\hat{\gamma}_\Lambda)}} e^{-E(\hat{\eta}) - C_\Lambda(\hat{\gamma})|\hat{\eta}|} \pi_\sigma(d\hat{\eta}) > 0. \]

For the integration by parts we need the following extra assumptions.

**Assumption 5** $V$ is once continuous differentiable for all $\hat{x}, \hat{y} \in \hat{X}$ with $x \neq y$, in symbols $V \in C^1(\Omega^{(2)}_X)$.

**Assumption 6** For all $\Lambda \in \mathcal{L}(\hat{X})$ we have
\[ \int_{\hat{X}} (\sup_{\hat{x} \in \Lambda} |\nabla^\hat{X} V(\hat{x}, \hat{y})|_{T_\hat{x} \hat{X}} \wedge 1) \sigma(d\hat{y}) < \infty. \]

Notice that this condition is satisfied if $\nabla^\hat{X} V(\hat{x}, \hat{y})$ is upper and lower regular, cf. Remark 5.2.

**Assumption 7** The second correlation measure $\rho^{(2)}_\mu$ corresponding to $\mu \in \mathcal{M}_1^{\lim}(\hat{\Gamma})$ satisfies the following condition: for all $\Lambda \in \mathcal{O}_c(\hat{X})$
\[ \int_{\hat{X}} \int_{\hat{X}} |\nabla^\hat{X} V(\hat{x}, \hat{y})|_{T_\hat{x} \hat{X}} \rho^{(2)}_\mu(d\hat{x}, d\hat{y}) < \infty. \]
**Assumption 8** The potential $V$ verify the following condition

$$\sup_{\hat{x}, \hat{y} \in \hat{X}} (|\nabla_{\hat{x}} V(\hat{x}, \hat{y})|_{T_{\hat{X}}}) e^{-V(\hat{x}, \hat{y})} < \infty.$$ 

**Corollary 5.5** Let $\mu \in M^1_{\text{lim}}(\hat{\Gamma})$ be a measure and $V$ be a potential such that the Assumptions 2-6 are valid. For any $\Lambda \in \mathcal{O}_c(\hat{X})$ with $\rho(\Lambda) = 0$ the sum

$$\sum_{\hat{x} \in \hat{\gamma}_{\Lambda^c}} \sup_{\hat{y} \in \Lambda} |\nabla_{\hat{x}} V(\hat{x}, \hat{y})|_{T_{\hat{X}}}$$

is $\mu$-a.s. finite and

$$W(\{\hat{x}\}, \gamma_{\Lambda^c}) = \sum_{\hat{y} \in \hat{\gamma}_{\Lambda^c}} V(\hat{x}, \hat{y})$$

is also $\mu$-a.s. absolutely convergent and differentiable with

$$\nabla_{\hat{x}} W(\{\hat{x}\}, \gamma_{\Lambda^c}) = \sum_{\hat{y} \in \hat{\gamma}_{\Lambda^c}} \nabla_{\hat{x}} V(\hat{x}, \hat{y})$$ (5.64)

and this last sum is also $\mu$-a.s. convergent.

The function $f : \lambda \mapsto \mathbb{R}$, $\hat{x} \mapsto e^{-E_\Lambda(\{\hat{x}\} \cup \gamma_{\Lambda^c})}$ is differentiable.

**Proof.** These results follow easily from Proposition 5.1 with

$$F(\hat{y}) = \mathbb{I}_{\Lambda^c}(\hat{y}) \left( \sup_{\hat{x} \in \Lambda} |V(\hat{x}, \hat{y})| + \sup_{\hat{x} \in \Lambda} |\nabla_{\hat{x}} V(\hat{x}, \hat{y})|_{T_{\hat{X}}} \right).$$

Moreover,

$$\sum_{\hat{x} \in \hat{\gamma}_{\Lambda^c}} \left( \sup_{\hat{x} \in \Lambda} |V(\hat{x}, \hat{y})| + \sup_{\hat{x} \in \Lambda} |\nabla_{\hat{x}} V(\hat{x}, \hat{y})|_{T_{\hat{X}}} \right) < \infty.$$

Using the uniform convergence of (5.64) the mapping $\hat{x} \mapsto E_\Lambda(\{\hat{x}\} \cup \gamma_{\Lambda^c})$ is differentiable and

$$\nabla_{\hat{x}} E_\Lambda(\{\hat{x}\} \cup \gamma_{\Lambda^c}) = \sum_{\hat{y} \in \hat{\gamma}_{\Lambda^c}} \nabla_{\hat{x}} V(\hat{x}, \hat{y}),$$

where the sum is $\mu$-a.s. absolutely convergent. \qed
Corollary 5.6 Let $\mu \in \mathcal{M}_{\text{fin}}(\Gamma)$ which fulfills Assumptions 4, 7 and $V$ a potential fulfilling Assumption 1-3 and 5, 6, 8. Let $\Lambda \in \mathcal{O}_c(\hat{X})$ with $\sigma(\partial \Lambda) = 0$. Then $r_{\hat{\gamma}}(\hat{\eta}) := \prod_{x \in \hat{\eta}} e^{-E_X(\hat{\eta} \cup \hat{\gamma}_\Lambda)}$ is differentiable for $\mu$-a.a. $\hat{\gamma}$ and belongs to $H^{1,1}(\Gamma^{(n)}_\Lambda, \sigma \otimes n)$.

**Proof.** By Corollary 5.5 for $\mu$-a.a. $\hat{\gamma}$ the function $E_X(\hat{\eta} \cup \hat{\gamma}_\Lambda)$ is differentiable with

$$\nabla \tilde{X} E_X(\{\tilde{x}\} \cup \hat{\gamma}_\Lambda) = \sum_{\tilde{y} \in \hat{\gamma}_\Lambda} \nabla \tilde{X} V(\tilde{x}, \tilde{y}),$$

and there exists a constant $C_\Lambda(\hat{\gamma})$ such that

$$|W(\{\tilde{x}\}, \hat{\gamma}_\Lambda)| \leq \sum_{\tilde{y} \in \hat{\gamma}_\Lambda} |V(\tilde{x}, \tilde{y})| \leq C_\Lambda(\hat{\gamma}).$$

Therefore, by stability $E_X(\hat{\eta} \cup \hat{\gamma}_\Lambda) \geq -Bn - C_\Lambda(\hat{\gamma}) n$. Hence $r_{\hat{\gamma}}(\hat{\eta})$ is differentiable and $\sigma$-a.s. strictly positive. For each $\hat{z} \in \hat{\eta}$ we have

$$\nabla \tilde{X} r_{\hat{\gamma}}(\hat{\eta}) = - \prod_{\tilde{x} \in \hat{\eta}} \rho(\tilde{x}) \nabla \tilde{X} (E(\hat{\eta}) + W(\hat{\eta}, \hat{\gamma}_\Lambda)) e^{-E_X(\hat{\eta} \cup \hat{\gamma}_\Lambda)}. \quad (5.65)$$

It remains to show that $\nabla \tilde{X} r_{\hat{\gamma}}(\hat{\eta}) \in L^1(\Gamma^{(n)}_\Lambda, \sigma \otimes n)$, to this end we compute separately the integral of the two summands on the right hand side of (5.65).

$$\int_{\Gamma^{(n)}_\Lambda} \left| \prod_{\hat{z} \in \hat{\eta}} \nabla \tilde{X} E(\hat{\eta}) e^{-E_X(\hat{\eta} \cup \hat{\gamma}_\Lambda)} \right| T_{2X} \sigma_n(d\hat{\eta})$$

$$\leq (n - 1) e^{Bn + C_\Lambda(\hat{\gamma})} n \sigma(\Lambda)^{n-2} \int_{\Lambda^2} \left| \nabla \tilde{X} V(\tilde{x}, \tilde{z}) \right| T_{2X} e^{-V(\tilde{x}, \tilde{z})} \sigma \otimes 2(d\tilde{x}, d\tilde{z})$$

$$\leq (n - 1) e^{Bn + C_\Lambda(\hat{\gamma})} n \sup_{\tilde{x}, \tilde{z} \in \Lambda} |\nabla \tilde{X} V(\tilde{x}, \tilde{z})| e^{-V(\tilde{x}, \tilde{z})}$$

which is finite by Assumption 8.

$$\int_{\Gamma^{(n)}_\Lambda} \left| \prod_{\hat{z} \in \hat{\eta}} \nabla \tilde{X} W(\hat{\eta}, \hat{\gamma}_\Lambda) e^{-E_X(\hat{\eta} \cup \hat{\gamma}_\Lambda)} \right| T_{2X} \sigma_n(d\hat{\eta})$$

$$\leq e^{Bn + C_\Lambda(\hat{\gamma})} n \sigma(\Lambda)^{n-1} \sum_{\tilde{x} \in \hat{\gamma}_\Lambda} \int_{\Lambda} \left| \nabla \tilde{X} V(\tilde{x}, \tilde{z}) \right| e^{-V(\tilde{x}, \tilde{z})} \sigma(d\tilde{z}).$$
We apply Proposition 5.1 for

\[ F(\hat{x}) = \mathbb{1}_{\Lambda_{\hat{x}}}(\hat{x}) \int_{\Lambda} \left| \frac{\nabla^X V(\hat{x}, \hat{z})}{T_{\hat{z}X}} \right| e^{-V(\hat{x}, \hat{z})} \sigma(d\hat{z}), \]

because firstly \( F(\hat{x}) = \infty \) implies \( \hat{z} \in \partial\Lambda \) and by Assumption 4 \( \rho^{(1)}(\partial\Lambda) = 0 \). Secondly

\[
\int_{\Lambda} \left| \frac{\nabla^X V(\hat{x}, \hat{z})}{T_{\hat{z}X}} \right| e^{-V(\hat{x}, \hat{z})} \sigma(d\hat{z}) \wedge 1
\]

and

\[
\sigma(\Lambda) e^{2B} \int_{\hat{z} \in \Lambda} \left| \frac{\nabla^X V(\hat{x}, \hat{z})}{T_{\hat{z}X}} \right| \sigma(d\hat{z}) < \infty
\]
is equivalent to Assumption 3.

5.2 Characterization via Radon-Nikodym derivative

Firstly, the Radon-Nikodym derivatives for a canonical marked Gibbs measure with respect to the diffeomorphism group are established.

**Theorem 5.7** Let \( \mu \in \mathcal{G}_c(\sigma, V) \cap \mathcal{M}_1^{\text{fin}}(\hat{\Gamma}) \) be a canonical marked Gibbs measure for a potential \( V \) fulfilling Assumption 2-4. Then \( \mu \) is \( \text{Diff}_{\text{large}}(\hat{X}) \)-quasi-invariant and

\[
\frac{d(\phi^*\mu)}{d\mu}(\hat{\gamma}) = \exp(-E_{\text{rel}}(\hat{\phi}^{-1}(\hat{\gamma}), \hat{\gamma})) \frac{d(\phi^*\pi_{\sigma})}{d\pi_{\sigma}}(\hat{\gamma}),
\]

where \( \hat{\phi} \in \text{Diff}_{\text{large}}(\hat{X}) \) and

\[
E_{\text{rel}}(\hat{\phi}^{-1}(\hat{\gamma}), \hat{\gamma}) := \sum_{\{\hat{x}, \hat{y}\} \in \hat{\gamma}} (V(\hat{\phi}^{-1}(\hat{x}), \hat{\phi}^{-1}(\hat{y})) - V(\hat{x}, \hat{y})).
\]

**Proof.** Let \( \hat{\phi} \in \text{Diff}_{\text{large}}(\hat{X}) \) be given and \( F : \Gamma \to \mathbb{R}_+^* \) a \( \mathcal{B}_\Lambda(\Gamma) \)-measurable function for a \( \Lambda \in \mathcal{O}_c(\hat{X}) \) with \( \sigma(\partial\Lambda) = 0 \) and w.l.o.g. \( \text{supp} \hat{\phi} \subset \Lambda \). Then by Definition 3.2 we have

\[
\int_{\hat{\Gamma}} F(\hat{\phi}(\hat{\gamma})) \mu(d\hat{\gamma}) = \int_{\hat{\Gamma}} \frac{K_{\hat{z}z_{\Lambda, \hat{\gamma}}(\hat{\gamma})}}{Z_{\Lambda}(\hat{\gamma})} \int_{\mathbb{R}^+(\Lambda)} F(\hat{\phi}(\hat{\eta})) e^{-E(\hat{\eta}) - W(\hat{\eta}, \hat{z}_{\Lambda, \hat{\gamma}})} \pi_{\sigma}(d\hat{\eta}) \mu(\hat{\gamma}).
\]
Note that $K_{\hat{z},\hat{z}}(\hat{\gamma}) = 1$ $\mu$-a.s. according Corollary 5.4. Applying the usual Radon-Nikodym theorem on the manifold $\Gamma^{(\hat{\gamma},\hat{\Lambda})}$ on the right hand side of (5.67) and taking into account that

$$
\frac{d(\hat{\phi}^*\sigma_{\hat{\Lambda}})}{d\sigma_{\hat{\Lambda}}}(\hat{\eta}) = \frac{d(\hat{\phi}^*\pi)}{d\pi}(\hat{\eta})
$$

we obtain

$$
\int_\hat{\Gamma} K_{\hat{z},\hat{z}}(\hat{\gamma}) \int_{\Gamma^{(\hat{\gamma},\hat{\Lambda})}} F(\hat{\eta}) \frac{1}{Z_{\hat{\Lambda}}(\hat{\gamma})} e^{-E(\hat{\phi}^{-1}(\hat{\eta})) - W(\hat{\phi}^{-1}(\hat{\eta}),\hat{\gamma}_{\hat{\Lambda}})} \frac{d(\hat{\phi}^*\pi)}{d\pi}(\hat{\eta}) \sigma_{\hat{\Lambda}}(d\hat{\eta}) \mu(\hat{\gamma}).
$$

(5.68)

Notice that

$$
E_{rel}(\hat{\phi}^{-1}(\hat{\eta} \cup \hat{\gamma}_{\hat{\Lambda}}), \hat{\eta} \cup \hat{\gamma}_{\hat{\Lambda}}) = E(\hat{\phi}^{-1}(\hat{\eta})) + W(\hat{\phi}^{-1}(\hat{\eta}), \hat{\gamma}_{\hat{\Lambda}}) - E(\hat{\eta}) - W(\hat{\eta}, \hat{\gamma})
$$

since all sums are absolutely convergent according Corollary 5.4 for $\mu$-a.a. $\hat{\gamma} \in \hat{\Gamma}$. Therefore, expression (5.68) is equal to

$$
\int_\Gamma \int_{\Gamma^{(\hat{\gamma},\hat{\Lambda})}} F(\hat{\eta}) e^{-E_{rel}(\hat{\phi}^{-1}(\hat{\eta} \cup \hat{\gamma}_{\hat{\Lambda}}), \hat{\eta} \cup \hat{\gamma}_{\hat{\Lambda}})} \frac{d(\hat{\phi}^*\pi)}{d\pi}(\hat{\eta}) \Pi_{\hat{\Lambda}}(d\hat{\eta}, \hat{\gamma}) \mu(d\gamma).
$$

The result now follows by Definition 3.2.

We proceed in order to show that condition (5.66) already characterizes canonical marked Gibbs measures.

**Theorem 5.8** Let $\mu \in M_{\text{im}}^1(\hat{\Gamma})$ fulfilling Assumptions 4 and a potential $V$ fulfilling Assumption 1-3. If for all $\hat{\phi} \in \text{Diff}_{\text{small}}(\hat{X})$ we have

$$
\frac{d(\hat{\phi}^*\mu)}{d\mu}(\hat{\gamma}) = \exp(-E_{rel}(\hat{\phi}^{-1}(\hat{\gamma}), \hat{\gamma})) \frac{d(\hat{\phi}^*\pi)}{d\pi}(\hat{\gamma}),
$$

(5.69)

then $\mu$ is a canonical marked Gibbs measure, i.e., $\mu \in G_c(\sigma, V)$.

**Proof.** $E_{rel}(\hat{\phi}^{-1}(\hat{\gamma}), \hat{\gamma}) := \sum_{(\hat{x},\hat{y}) \in \hat{\gamma}} V(\hat{\phi}^{-1}(\hat{x}), \hat{\phi}^{-1}(\hat{y})) - V(\hat{x}, \hat{y})$ is well defined and the series is $\mu$-a.s. absolutely convergent according to Corollary 5.4.

Let $\hat{\phi}$ be a diffeomorphism from $\text{Diff}_{\text{small}}(\hat{X})$ and choose $\Lambda \in \mathcal{O}_c(\hat{X})$ connected with $\sigma(\partial \Lambda) = 0$ such that $\text{supp}\hat{\phi} \subset \Lambda$. Take a $F = F_1 \cdot F_2$ where
$F_1$ is $\mathcal{B}_\Lambda(\hat{\Gamma})$-measurable and $F_2$ is $\mathcal{B}_{\Lambda^c}(\hat{\Gamma})$-measurable. If we denote by $\mu_\Lambda$ the conditional probability measure of $\mu$ with respect to $\mathcal{F}_{\Lambda^c}$, then we have to show that $\mu_\Lambda$ is equal to (3.19) $\mu$-a.s. Hence, using the definition of the conditional probability we can write

$$\int_\hat{\Gamma} F(\hat{\phi}(\hat{\gamma})) \mu(d\hat{\gamma}) = \int_\Gamma \int_{\Gamma_\Lambda(\hat{\gamma})} F_2(\hat{\gamma}) F_1(\hat{\gamma}) \mu_\Lambda(d\hat{\eta}, \hat{\gamma}) \mu(d\hat{\eta}).$$

On the other hand we have

$$\int_\hat{\Gamma} F(\hat{\phi}(\hat{\gamma})) \mu(d\hat{\gamma}) = \int_\hat{\Gamma} F_2(\hat{\gamma}) \int_{\Gamma_\Lambda(\hat{\gamma})} F_1(\hat{\gamma}) e^{-E_{\mu(\hat{\eta})}(\hat{\phi}(\hat{\gamma}))} \frac{d(\hat{\phi}^*\pi_\sigma)}{d\pi_\sigma}(\hat{\eta}) \mu_\Lambda(d\hat{\eta}, \hat{\gamma}) \mu(d\hat{\eta}).$$

As $\text{Diff}_{\text{small}}(\hat{X})$ is countable and $\mathcal{B}_\Lambda(\Gamma)$ is countably generated for $\mu$-a.a. $\hat{\gamma}$ the following equality holds for all $F_3 \in L^0(\Gamma, \mathcal{B}(\Gamma))$

$$\int_{\Gamma_\Lambda(\hat{\gamma})} F_3(\hat{\phi}(\hat{\eta})) \mu_\Lambda(d\hat{\eta}, \hat{\gamma}) = \int_{\Gamma_\Lambda(\hat{\gamma})} F_3(\hat{\eta}) e^{-E_{\mu(\hat{\eta})}(\hat{\phi}(\hat{\gamma}))} \frac{d(\hat{\phi}^*\pi_\sigma)}{d\pi_\sigma}(\hat{\eta}) \mu_\Lambda(d\hat{\eta}, \hat{\gamma}).$$

Now we apply Lemma 4.10 for $r_{\gamma}(\hat{\eta}) := e^{-E_{\Lambda}(\hat{\eta})} \prod_{x \in \hat{\gamma}} \rho(\hat{x})$. According to Corollary 5.4

$$E_{\Lambda}(\hat{\eta} \cup \hat{\gamma}_{\Lambda^c}) := \sum_{x \in \hat{\eta}} \sum_{y \in \hat{\gamma}_{\Lambda^c}} V(\hat{x}, \hat{y})$$

is for $\mu$-a.a. $\gamma$ absolutely convergent, hence for $\mu$-a.a. $\hat{\gamma}$ the function $r_{\gamma}(\hat{\eta}) > 0$ for all $\hat{\eta} \in \Gamma_\Lambda(\hat{\gamma})$ and

$$\frac{r_{\gamma}(\hat{\phi}^{-1}(\hat{\eta}))}{r_{\gamma}(\hat{\gamma})} \prod_{x \in \eta} J_{\phi}^{-1}(\hat{x}) = e^{-E_{\mu(\hat{\eta})}(\hat{\phi}(\hat{\gamma}))} \frac{d(\hat{\phi}^*\pi_\sigma)}{d\pi_\sigma}(\hat{\eta}).$$

Thus the measure $\mu_\Lambda$ is of the form

$$\mu_\Lambda(d\hat{\eta}, \hat{\gamma}) = k \mathbb{I}_{\{N_{\Lambda} = N_{\Lambda}(\hat{\gamma})\}}(\hat{\eta}) e^{-E_{\Lambda}(\hat{\eta})} \sigma_{\hat{\gamma}_{\Lambda^c}}(d\hat{\eta}).$$

Since $\mu_\Lambda$ is a probability measure on $\Gamma_\Lambda$ we have $k = (Z_\Lambda(\hat{\gamma}))^{-1}$ and $0 < Z_\Lambda(\hat{\gamma}) < \infty$ (cf. Corollary 5.4). Summarizing, $\mu_\Lambda$ is just

$$\mu_\Lambda(d\hat{\eta}, \hat{\gamma}) = \frac{1}{Z_\Lambda(\hat{\gamma})} \mathbb{I}_{\{N_{\Lambda} = N_{\Lambda}(\hat{\gamma})\}}(\hat{\eta}) e^{-E_{\Lambda}(\hat{\eta})} \sigma_{\hat{\gamma}_{\Lambda^c}}(d\hat{\eta})$$

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which coincides with (3.19). Thus $\mu \in \mathcal{G}_c(\sigma, V)$ by Definition 3.2.

5.3 Characterization via integration by parts

We now give another characterization of canonical marked Gibbs measures via integration by parts. Before let us derive the integration by parts formula for such measures.

**Theorem 5.9** Let $\mu \in \mathcal{G}_c(\sigma, V) \cap \mathcal{M}^1_{\text{fin}}(\hat{\Gamma})$ be a given canonical marked Gibbs measure which fulfills Assumption 4, 7 and a potential $V$ which fulfills Assumption 2, 3, 5, and 7. Then for all vector fields $\hat{v} \in \text{Vect}_\text{max}(\hat{X})$ and all $F \in \mathcal{F}\mathcal{C}_b^\infty(D, \hat{\Gamma})$ we have

$$
\int_{\hat{\Gamma}} \langle \nabla F(\hat{\gamma}), \text{Exp}\hat{v}(\hat{\gamma}) \rangle_{T, \hat{\Gamma}} \mu(d\hat{\gamma})
= -\int_{\hat{\Gamma}} F(\hat{\gamma}) \text{div} \text{Exp}\hat{v}(\hat{\gamma}) + \langle \text{Exp}\hat{v}(\hat{\gamma}), \beta(\hat{\gamma}) + B^V(\hat{\gamma}) \rangle_{T, \hat{\Gamma}} \mu(d\hat{\gamma}),
$$

where $\beta$ is defined as in (4.53) and $B^V$ is given by

$$
B^V(\hat{\gamma}) = (B^V_2(\hat{\gamma}))_{\hat{x} \in \hat{\gamma}} = \left( -\sum_{\hat{y} \in \hat{\gamma}\setminus\{\hat{x}\}} \nabla^X \hat{v}(\hat{x}, \hat{y}) \right)_{\hat{x} \in \hat{\gamma}}
$$

and this last sum is $\mu$-a.s. absolutely convergent.

**Proof.** Let $\hat{v} \in \text{Vect}_\text{small}(\hat{X})$ and $F \in \mathcal{F}\mathcal{C}_b^\infty(D, \hat{\Gamma})$ be given. Then, there exists $\Lambda \in \mathcal{O}_c(\hat{X})$ with $\sigma(\partial\Lambda) = 0$ such that $\text{supp}\hat{v} \subset \Lambda$ and $F$ is $\mathcal{B}_\Lambda(\hat{\Gamma})$-measurable. First we prove that all integrals in (5.70) do exists. Denote by $C$ the upper bound of the functions $|F|$ and $|\nabla F|$. Then we have using Assumption 4

$$
\int_{\hat{\Gamma}} |\langle \nabla F(\hat{\gamma}), \text{Exp}\hat{v}(\hat{\gamma}) \rangle_{T, \hat{\Gamma}}| \mu(d\hat{\gamma}) \leq C \int_{\hat{\Gamma}} \sum_{\hat{x} \in \hat{\gamma}} |\hat{v}(\hat{x})|_{T, \hat{X}} \mu(d\hat{x})
$$

$$
= C \int_{\hat{X}} |\hat{v}(\hat{x})|_{T, \hat{X}} \rho^{(1)}(d\hat{x})
\leq C C_1 \sup_{\hat{x} \in \hat{X}} |\hat{v}(\hat{x})|_{T, \hat{X}} \sigma(\Lambda).
$$

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On the other hand, the right hand side of (5.70) may be estimated as follows

\[
\int_\Gamma \left[ F(\hat{\gamma}) \left[ \text{div}^\tilde{\gamma} \text{Exp}^\hat{\gamma} + \langle \text{Exp}^\hat{\gamma}, \beta(\hat{\gamma}) + B^V(\hat{\gamma}) \rangle_{T_\gamma \Gamma} \right] \right] \mu(d\hat{\gamma}) \\
\leq C \int_\Gamma \sum_{x \in \hat{\gamma}} \left[ \left| \text{div}^\tilde{\gamma} \hat{v}(\hat{x}) \right| + \left| \langle \hat{v}(\hat{x}), \beta_\sigma(\hat{x}) \rangle_{T_\gamma \hat{\Gamma}} \right| \right] \mu(d\hat{\gamma}) \\
+ \sum_{y \in \hat{\gamma} \setminus \{\hat{x}\}} \left| \langle \hat{v}(\hat{x}), \nabla^\tilde{\gamma} V(\hat{x}, \hat{y}) \rangle_{T_\gamma \hat{\Gamma}} \right| \mu(d\hat{\gamma}) \\
= C \int_{\hat{\Gamma}} \left| \text{div}^\tilde{\gamma} \hat{v}(\hat{x}) \right|_{T_\gamma \hat{\Gamma}} + \left| \langle \hat{v}(\hat{x}), \beta_\sigma(\hat{x}) \rangle_{T_\gamma \hat{\Gamma}} \rho^{(1)}(d\tilde{x}) \right|
+ 2C \int_{\hat{\Gamma}} \left| \langle \hat{v}(\hat{x}), \nabla^\tilde{\gamma} V(\hat{x}, \hat{y}) \rangle_{T_\gamma \hat{\Gamma}} \rho^{(2)}(d\tilde{x}, d\tilde{y}) \right|
\leq C \sup_{\hat{x} \in \hat{\Gamma}} \left( \left| \text{div}^\tilde{\gamma} \hat{v}(\hat{x}) \right|_{T_\gamma \hat{\Gamma}} + \left| \langle \hat{v}(\hat{x}) \rangle_{T_\gamma \hat{\Gamma}} \right| \right)
\cdot \left( C_1 \sigma(\Lambda) + C_1 \int_{\hat{\Gamma}} \left| \beta_\sigma(\hat{x}) \right|_{T_\gamma \hat{\Gamma}} \sigma(d\tilde{x}) + \int_{\Lambda} \int_{\hat{\Gamma}} \left| \nabla^\tilde{\gamma} V(\hat{x}, \hat{y}) \right|_{T_\gamma \hat{\Gamma}} \rho^{(2)}(d\tilde{x}, d\tilde{y}) \right). 
\]

All quantities on the proceeding estimate are finite because of the definition of \( \text{Vect}_{\text{large}}(\hat{\Gamma}) \), \( \beta_\sigma \in L^1_{\text{loc}}(\hat{\Gamma}, \sigma) \), and Assumption 7. This proves the existence of the integrals in (5.70).

We now proceed in order to derive the integration by parts formula (5.70). According to Definition 3.2 we have

\[
\int_{\hat{\Gamma}} \left( \nabla^\tilde{\gamma} F(\hat{\gamma}), \text{Exp}^\hat{\gamma} \right)_{T_\gamma \hat{\Gamma}} \mu(d\hat{\gamma}) \\
= \int_{\hat{\Gamma}} \int_{\Gamma^{[i_\Lambda]}} \left( \nabla^\tilde{\gamma} F(\hat{\eta}), \text{Exp}^\hat{\gamma} \right)_{T_\gamma \Gamma} \Pi^\hat{\gamma}(d\hat{\gamma}, \hat{\gamma}) \mu(d\hat{\gamma}) \\
= \int_{\hat{\Gamma}} \frac{K_{\hat{\gamma}, \sigma(\Lambda)}(\hat{\gamma})}{Z(\hat{\gamma})} \left( \int_{\hat{\Gamma}^{[i_\Lambda]}} \sum_{i=1}^{[i_\Lambda]} \left( \nabla^\tilde{\gamma} F(\{\hat{x}\}_{1}^{[i_\Lambda]}, \hat{v}(\hat{\gamma}), T_\gamma \Gamma) \\ e^{-E_{\Lambda}(\{\hat{x}\}_{1}^{[i_\Lambda]} \cup \hat{\gamma})} \sigma(d\hat{x}) \right) \mu(d\hat{\gamma}) \right). 
\]

Note that, according to Corollary 5.6, \( e^{-E_{\Lambda}(\{\hat{x}\}_{1}^{[i_\Lambda]} \cup \hat{\gamma})} \in H^{1,1}(\hat{\Gamma}, (\sigma)^{[i_\Lambda]}) \) and \( > 0 \) for \( \mu \)-a.a. \( \hat{\gamma} \). Applying the usual integration by parts formula on the manifold \( \Gamma^{[i_\Lambda]} \) to (5.72) we obtain having in mind that \( \nabla^\tilde{\gamma} E_{\Lambda}(\{\hat{x}\}_{1}^{[i_\Lambda]} \cup \hat{\gamma}) \).
\[ \hat{\gamma}_{\Lambda^c} = B^V_{\hat{x}}(\{\hat{x}\}^{|\hat{\alpha}|}_1 \cup \hat{\gamma}_{\Lambda^c}) \]

\[ - \int_{\Gamma} \frac{K_{\hat{x}}(\hat{x})}{Z_{\Lambda}(\hat{\gamma})} \int_{\Lambda^{|\hat{\alpha}|}_1} F(\{\hat{x}\}^{|\hat{\alpha}|}_1) \sum_{i=1}^{|\hat{\alpha}|} \left[ \text{div}^{\hat{X}} \hat{v}(\hat{x}_i) + \langle \hat{v}(\hat{x}), \beta \rangle \right] \]

\[ \cdot + B^V_2(\{\hat{x}\}^{|\hat{\alpha}|}_1 \cup \hat{\gamma}_{\Lambda^c})_{\hat{\gamma}} \int_{\hat{x}, \hat{X}} e^{E_{\Lambda}(\{\hat{x}\}^{|\hat{\alpha}|}_1 \cup \hat{\gamma}_{\Lambda^c})} \sigma(d\hat{x})^{|\hat{\alpha}|}_1 \mu(d\hat{\gamma}) \]

\[ = - \int_{\Gamma} \int_{\Gamma^{(\hat{\alpha})}_\Lambda} F(\hat{\eta}) \text{div}^{\hat{\Gamma}} \text{Exp}(\hat{\eta})_T \mu(d\hat{\gamma}) \]

\[ - \int_{\Gamma} \int_{\Gamma^{(\hat{\alpha})}_\Lambda} F(\hat{\eta}) \langle \text{Exp}(\hat{\eta}), \beta(\hat{\eta}) \rangle + B^V(\hat{\eta} \cup \hat{\gamma}_{\Lambda^c})_{T_{\hat{\eta}}} \Gamma^{(\hat{\alpha})}_\Lambda \mu(d\hat{\gamma}), \]

where \( B^V \) is given by (5.71) and \( \beta \) by (4.53). The result of the theorem follows applying Definition 3.2 once more.

As a type of converse of Theorem 5.9 we proof

**Theorem 5.10** Let \( \mu \in \mathcal{M}_1(\hat{\Gamma}) \) which fulfills Assumptions 4, 7 and a potential \( V \) which fulfills Assumption 1-3, 5-8. Suppose that for all \( \hat{v} \in \text{Vect}_{\text{small}}(\hat{X}) \) we have the following integration by parts formula for \( F \in \mathcal{F}C^\infty_p(\hat{\Gamma}, B(\hat{\Gamma})) \)

\[ \int_{\hat{\Gamma}} \langle \nabla^{\hat{\Gamma}} F(\hat{\gamma}), \text{Exp}(\hat{\gamma}) \rangle_{T_{\hat{\Gamma}}} \mu(d\hat{\gamma}) \]

\[ = - \int_{\hat{\Gamma}} F(\hat{\gamma}) \left[ \text{div}^{\hat{\Gamma}} \text{Exp}(\hat{\gamma}) + \langle \text{Exp}(\hat{\gamma}), \beta(\hat{\gamma}) \rangle + B^V(\hat{\gamma}) \rangle_{T_{\hat{\gamma}}} \hat{\Gamma} \right] \mu(d\hat{\gamma}), \]

where the vectors fields \( \beta(\hat{\gamma}) \) and \( B^V(\hat{\gamma}) \) are defined in (4.53) and (5.71), respectively. Then \( \mu \) is a canonical marked Gibbs measure.

**Proof.** As in the proof of Theorem 5.9 we may show that the integrals exists. Take a local set \( \Lambda \in \mathcal{O}_c(\hat{X}) \) and \( \sigma(\partial \Lambda) = 0 \) and a vector field \( \hat{v} \) from \( \text{Vect}_{\text{small}}(\Lambda) \). Consider cylinder function \( F_1 \in \mathcal{F}C^\infty_p(D, \hat{\Gamma}, B(\hat{\Gamma})) \) and \( F_2 \in \mathcal{F}C^\infty_p(D, \hat{\Gamma}, B(\hat{\Gamma})) \). Then we have

\[ \int_{\Gamma}^{(\hat{\alpha})}_A \langle \nabla^{\hat{\Gamma}} F_1(\hat{\eta}), \text{Exp}(\hat{\eta}) \rangle_{T_{\hat{\eta}} \Gamma^{(\hat{\alpha})}_A} \mu_A(d\hat{\eta}, \hat{\gamma}) \]

\[ = \int_{\Gamma}^{(\hat{\alpha})}_A F_1(\hat{\eta}) \left[ \text{div}^{\hat{\Gamma}} \text{Exp}(\hat{\eta}) + \langle \text{Exp}(\hat{\eta}), \beta(\hat{\eta}) \rangle - B^V(\hat{\eta} \cup \hat{\gamma}_{\Lambda^c}) \rangle_{T_{\hat{\eta}} \Gamma^{(\hat{\alpha})}_A} \mu_A(d\hat{\eta}, \hat{\gamma}). \]

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According to Corollary 5.6 the function \( r\hat{\gamma}(\hat{\eta}) := \prod_{\hat{x} \in \hat{\eta}} e^{-E_\Lambda(\hat{\eta}, \hat{\gamma}_\Lambda \sigma)} \in H^{1,1}(\Gamma_n, \sigma_n) \) and > 0 for \( \mu \)-a.a. \( \hat{\gamma} \). Applying Lemma 4.17 we obtain
\[
\mu_\Lambda(d\hat{\eta}, \hat{\gamma}) = \frac{1}{Z_\Lambda(\hat{\gamma})} \mathbb{1}_{\{N_\Lambda=N_\Lambda(\hat{\gamma})\}}(\hat{\eta}) e^{-E_\Lambda(\hat{\eta}, \hat{\gamma} \Lambda \sigma)} \sigma_{\hat{\gamma}_\Lambda}(d\hat{\eta}).
\]
Since \( \mu \)-a.s. 0 < \( Z_\Lambda(\hat{\gamma}) \leq \Xi_\Lambda(\hat{\gamma}) < \infty \) we have the equality
\[
\mu_\Lambda(d\hat{\eta}, \hat{\gamma}) = \mathbb{1}_{\{N_\Lambda=N_\Lambda(\hat{\gamma})\}}(\hat{\eta}) \frac{1}{Z_\Lambda(\hat{\gamma})} e^{-E_\Lambda(\hat{\eta}, \hat{\gamma} \Lambda \sigma)} \sigma_{\hat{\gamma}_\Lambda}(d\hat{\eta}).
\]
Therefore \( \mu \) is a canonical marked Gibbs measure. \( \blacksquare \)
6 Ergodicity

A canonical Gibbs measure is ergodic iff it is extreme, see Theorem 6.4. Ergodicity is a necessary and sufficient condition that the Vershik-Gelfand-Graev type representation of measures on configuration space $\Gamma$ is irreducible, see e.g. [GGV75] and [Ism96] and Theorem 6.5 for more details.

Let us recall that $G_c(\sigma, V)$ is the set of all probability measures $\mu$ on $(\hat{\Gamma}, B(\hat{\Gamma}))$ such that $\mu = \mu \Pi_\Lambda^\sigma$ for all $\Lambda \in \mathcal{L}(\hat{X})$, cf. Definition 3.2. It follows immediately that $G_c(\sigma, V)$ is a convex set, i.e., if $\mu_1, \mu_2 \in G_c(\sigma, V)$ and $0 \leq \alpha \leq 1$, then $\alpha \mu_1 + (1-\alpha) \mu_2 \in G_c(\sigma, V)$. A measure $\mu \in G_c(\sigma, V)$ is a trivial measure whenever there is $\mu_1, \mu_2 \in G_c(\sigma, V)$ and $0 \leq \alpha \leq 1$ with $\mu = \alpha \mu_1 + (1-\alpha) \mu_2$, then $\mu = \mu_1 = \mu_2$. The tail field $\sigma$-algebra $\mathcal{F}_\infty(\hat{\Gamma})$ is defined by

$$\mathcal{F}_\infty(\hat{\Gamma}) := \bigcap_{\Lambda \in \mathcal{L}(\hat{X})} \mathcal{F}_{\Lambda^\sigma}(\hat{\Gamma}).$$

The following results from [Pre80] are used in this section.

**Lemma 6.1** Let $\mu, \mu' \in G_c(\sigma, V)$ and let $F : \hat{\Gamma} \to \mathbb{R}^+$ be a $\mathcal{B}(\Gamma)$-measurable function with $\int_{\hat{\Gamma}} F(\hat{\gamma}) \mu(d\hat{\gamma}) = 1$.

(i) $\mu$ is extreme iff $\mu$ is trivial on $\mathcal{F}_\infty(\hat{\Gamma})$, i.e., $\mu(B)$ is either 0 or 1 for each $B \in \mathcal{F}_\infty(\hat{\Gamma})$.

(ii) $F \mu \in G_c(\sigma, V)$ iff $\mathbb{E}[F|\mathcal{F}_\infty(\hat{\Gamma})] = F$ $\mu$-a.s..

(iii) If $\mu \neq \mu'$ then $\mu \perp \mu'$, i.e., there exists a $B \in \mathcal{F}_\infty(\Gamma)$ with $\mu(B) = 1$ and $\mu'(B) = 0$.

**Proof.** i) is Theorem 5.1. of [Pre80] and (ii) is just Lemma 5.2. of [Pre80].

We call a measure $\mu \in \mathcal{M}^1_{\text{im}}(\hat{\Gamma})$ admissible if $\mu$ fulfills Assumption 4, i.e., $\mu \in \mathcal{M}^1_{\text{im}}(\hat{\Gamma})$ and exists $C_1 > 0$ such that

$$0 \leq \frac{d\rho_\mu^{(1)}}{d\sigma} \leq C_1.$$

The set of all admissible measures is convex and set of all admissible canonical Gibbs measures $G_{c,a}(\sigma, V)$ is a face of $G_c(\sigma, V)$. In fact
Lemma 6.2 The extreme measure in the set of admissible canonical Gibbs measures are exactly the admissible extreme measures in the set of canonical Gibbs measure, in symbols \( \text{ext} \left( G_{c,a}(\sigma, V) \right) = \text{ext} \left( G_c(\sigma, V) \right) \cap G_{c,a}(\sigma, V) \).

Proof. If \( \mu \) is extreme in \( G_c(\sigma, V) \) then it is also extreme in \( G_{c,a}(\sigma, V) \). If \( \mu \) is an extreme measure in \( G_{c,a}(\sigma, V) \) and there exists (in general non-admissible) canonical Gibbs measures \( \mu_1, \mu_2 \) such that \( \mu = \frac{1}{2} \mu_1 + \frac{1}{2} \mu_2 \). We want to show that the \( \mu_i \) are in fact admissible. For every \( \Lambda \in \mathcal{L}(X) \) and \( n \in \mathbb{N} \) we have

\[
\int_{\Gamma} |\gamma_\Lambda|^n \mu_i(d\gamma) \leq 2 \int_{\Gamma} |\gamma_\Lambda|^n \mu(d\gamma) < \infty.
\]

Hence \( \mu_i \in \mathcal{M}^1(\hat{\Gamma}) \). Also \( \rho_\mu = \frac{1}{2} \rho_{\mu_1} + \frac{1}{2} \rho_{\mu_2} \) and therefore \( \rho_{\mu_i} \ll \rho_\mu \) and

\[
0 \leq \frac{d\rho_{\mu_i}}{d\sigma} \leq 2 \frac{d\rho_\mu}{d\sigma} \leq 2C_1.
\]

Thus \( \mu_i \) is admissible. Due to the extremity of \( \mu \) in \( G_{c,a}(\sigma, V) \) is \( \mu = \mu_1 = \mu_2 \) and \( \mu \) is also extreme in \( G_c(\sigma, V) \).

The following lemma contains the part of the proof specific for the relation between Gibbs measures and the diffeomorphism group which is based on the characterization theorem via Radon-Nikodym derivative. The main result follows then by general considerations for Gibbs measures, cf. e.g. [Pre80].

Lemma 6.3 Let \( \mu \in \mathcal{M}^1(\hat{\Gamma}) \cap G_c(\sigma, V) \) fulfilling Assumptions 4 for a potential \( V \) fulfilling Assumption 1-3. Let \( F : \hat{\Gamma} \rightarrow \mathbb{R}_+ \) be a measurable bounded function with \( \int_{\Gamma} F(\gamma) \mu(d\gamma) = 1 \) such that \( F \circ \phi = F \) \( \mu \)-a.s. for all \( \phi \in \text{Diff}_{\text{small}}(X) \). Then \( \nu := F \mu \) is also a canonical marked Gibbs measure.

Proof. Let \( G : \hat{\Gamma} \rightarrow \mathbb{R}_+ \) be another measurable bounded function. Then for \( \phi \in \text{Diff}_{\text{small}}(X) \) it follows that

\[
\int_{\Gamma} G(\phi(\gamma)) \nu(d\gamma) = \int_{\Gamma} G(\gamma) F(\phi^{-1}(\gamma)) d(\phi^* \mu)(\gamma) = \int_{\Gamma} G(\gamma) \frac{d(\phi^* \mu)}{d\mu}(\gamma) \nu(d\gamma).
\]

Therefore \( \nu \)-a.s. we have \( \frac{d(\phi^* \mu)}{d\sigma}(\gamma) = \frac{d(\phi^* \nu)}{d\nu}(\gamma) \). Furthermore, for any measurable \( H : \Gamma \rightarrow \mathbb{R}_+ \) function and for \( C := \sup_{\gamma \in \Gamma} |F(\gamma)| \) we have

\[
\int_{\Gamma_0} H(\gamma) \rho_\nu(d\gamma) \leq \int_{\Gamma} (KH)(\gamma) \nu(d\gamma) = \int_{\Gamma} (KH)(\gamma) F(\gamma) \mu(d\gamma) \leq C \int_{\Gamma} (KH)(\gamma) \mu(d\gamma) \leq C \int_{\Gamma_0} H(\gamma) \rho_\mu(d\gamma).
\]
which implies that $\rho_\nu(d\tilde{\eta}) \leq C \rho_\mu(d\tilde{\eta})$. Hence with $\mu$ also $\nu$ fulfills the assumption of Theorem 5.8 (characterization theorem via Radon-Nikodym derivative) and we deduce that $\nu$ is a canonical marked Gibbs measure.

We are now prepared to prove the main results of this section.

**Theorem 6.4** Let $\mu \in \mathcal{M}_1^{\text{fm}}(\hat{\Gamma}) \cap \mathcal{G}_c(\sigma,V)$ fulfilling Assumptions 4 for a potential $V$ fulfilling Assumption 1-3. $\mu$ is extreme iff it is $\text{Diff}_{\text{small}}(\hat{X})$-ergodic.

**Proof.** Assume that $\mu$ is an extreme measure. Let $F : \hat{\Gamma} \to \mathbb{R}^+$ be a measurable bounded function such that $F \circ \phi = F \mu$-a.s. for all $\phi \in \text{Diff}_{\text{small}}(\hat{X})$. Without lost of generality we may assume that $\int_{\hat{\Gamma}} F(\tilde{\gamma}) \mu(d\tilde{\gamma}) = 1$. Then we have to prove that $F$ is constant $\mu$-a.s. According to Lemma 6.3 it is $F \mu \in \mathcal{G}_c(\sigma,V)$ and applying Lemma 6.1 (ii) we obtain

$$\mathbb{E}_\mu(F|\mathcal{F}_\infty(\hat{\Gamma})) = F, \mu - \text{a.s.}$$

On the other hand according to Lemma 6.1 (i) we know that $\mu$ is trivial on $\mathcal{B}_\infty(\hat{\Gamma})$, hence for any $A \in \mathcal{B}(\mathbb{R}_+)$ we have

$$\mu(\{\tilde{\gamma} \in \hat{\Gamma} | F(\tilde{\gamma}) \in A\}) \in \{0,1\}.$$  

This implies that $F$ is constant $\mu$-a.s. Hence $\mu$ is $\text{Diff}_{\text{small}}(\hat{X})$-ergodic.

Conversely, assume that $\mu$ is $\text{Diff}_{\text{small}}(\hat{X})$-ergodic and there exists $\mu_1, \mu_2 \in \mathcal{G}_c(\sigma,V)$ such that $\mu = \frac{1}{2}(\mu_1 + \mu_2)$. Thus $\mu_1 \ll \mu$ and there exists a measurable function $F : \hat{\Gamma} \to \mathbb{R}^+$ with $\int_{\hat{\Gamma}} F(\tilde{\gamma}) \mu(d\tilde{\gamma}) = 1$ such that $\mu_1 = F \mu$. It follows from Lemma 6.1 (ii) that $\mathbb{E}_\mu(F|\mathcal{F}_\infty(\hat{\Gamma})) = F$ $\mu$-a.s. and, hence for any $\phi \in \text{Diff}_{\text{small}}(\hat{X})$ we obtain

$$F \circ \phi = \mathbb{E}_\mu(F|\mathcal{F}_\infty(\hat{\Gamma})) \circ \phi = \mathbb{E}_\mu(F|\mathcal{F}_\infty(\hat{\Gamma})) = F.$$  

In the second equality we use the fact that $\mathbb{E}_\mu(F|\mathcal{F}_\infty(\hat{\Gamma}))$ is $\mathcal{F}_{\Lambda'}(\hat{\Gamma})$-measurable for a certain $\Lambda \in \mathcal{L}(\hat{X})$ such that $\text{supp}\phi \subset \Lambda$. Since $\mu$ is $\text{Diff}_{\text{small}}(\hat{X})$-ergodic it implies that $F$ is constant $\mu$-a.s. Therefore $\mu_1 = \mu_2 = \mu$ and this proves that $\mu$ is extreme.

We are now ready to announce the result concerning the irreducibility of the unitary representation $V_\mu$ of the group $\text{Diff}_{\text{max}}(\hat{X})$ associated with $\mu \in \mathcal{G}_c(\sigma,V)$. The proof is a consequence of the results in this section and Theorem 1, §3 in [GGV75] or Corollary 28.1, Chapter 5 of [Ism96].
**Theorem 6.5** Let $\mu \in \mathcal{M}^{1}_{\text{im}}(\Gamma) \cap \mathcal{G}_c(\sigma, V)$ be an admissible canonical Gibbs measure on the (unmarked) configuration space $\Gamma$ fulfilling Assumptions 4 for a potential $V$ fulfilling Assumption 1-3. Then the unitary representation

$$(V_\mu(\phi)F)(\gamma) := \sqrt{\frac{d\phi^* \mu}{d\mu}(\gamma)}F(\phi^{-1}(\gamma)), \quad F \in L^2(\Gamma, \mu), \phi \in \text{Diff}^0_0(X) \quad (6.74)$$

is irreducible iff $\mu$ is extreme.

**Proof.** We use Theorem 6.4 and the fact that the irreducibility of $V_\mu$ is equivalent to the $\text{Diff}^0_0(X)$-ergodicity of $\mu$. That ergodic measures on configuration spaces leads to irreducible representation is proven in Theorem 1, §3 in [GGV75] or Corollary 28.1, Chapter 5 of [Ism96]. The converse holds generally: Let $F$ be a $\mu$-a.s. $\text{Diff}^0_0(X)$-invariant set. Then the space $L^2(F, \mu)$ is invariant under the action of $V_\mu$ and hence by irreducibility $\mu(F) = 0, 1$. ■
7 Equivalence of Ensembles

We want to consider equivalence of ensembles in the following sense, see e.g., [Pre79, Section 2, 3]

\[ \text{ext} \mathcal{G}_c(\sigma, V) = \{\delta_{\emptyset}\} \cup \bigcup_{z>0} \text{ext} \mathcal{G}_{gc}(z\sigma, V). \]  \hfill (7.75)

This tells us that the physical state of the system (extreme measures) are coinciding for the canonical and the grand canonical case. In particular, a measure in \( \mathcal{G}_{gc}(z\sigma, V) \) is \( \text{Diff}_0(\hat{X}) \)-ergodic iff it is extreme in \( \mathcal{G}_{gc}(z\sigma, V) \).

Let \( \mu \) be a measure on \( (\hat{\Gamma}, \mathcal{B}(\hat{\Gamma})) \) satisfying Assumptions 3, 2, and 4. Then it follows from Corollary 5.4 that there exists a set \( F \in \mathcal{B}(\hat{\Gamma}) \) of full \( \mu \)-measure such for all \( \hat{\gamma} \in F \) and any finite volume \( \Lambda \in \mathcal{O}_c(\hat{X}) \) with \( \sigma(\partial\Lambda) = 0 \) (Note not for all \( L(\hat{X}) \)) we have

\[ 0 < Z_{\Lambda}(\hat{\gamma}) \leq \Xi_{\Lambda}(\hat{\gamma}) < \infty, \quad W(\hat{\eta}, \hat{\gamma}_{\Lambda'}) \leq |\hat{\eta}|C_{\Lambda}(\hat{\gamma}) \]

for a constant \( C_{\Lambda}(\hat{\gamma}) \). Before we prove the main result of this section we state a useful Lemma.

**Lemma 7.1** Let \((X, \mathcal{B}, \sigma)\) be a measure space and \( f : X \to \mathbb{R}_+ \cup \{0\} \) a non-negative function such that \( \sigma(\{f > 0\}) > 0 \). Then we have \( \int_X f(x)\sigma(dx) > 0 \).

**Proof.** For each \( n \in \mathbb{N} \) we define the set \( A_n := \{f \geq \frac{1}{n}\} \). It is clear that \( A_n \) is an increasing sequence of subsets in \( X \) such that \( A_n \uparrow \{f > 0\} \). Therefore by the monotonicity of the measure \( \sigma \) we have \( \sigma(A_n) \uparrow \sigma(\{f > 0\}) > 0 \). Hence there exists an \( N \in \mathbb{N} \) such that \( \sigma(\{f > \frac{1}{N}\}) > 0 \) and we have

\[ \int_X f(x)\sigma(dx) \geq \frac{1}{N}\sigma\left(\left\{f > \frac{1}{N}\right\}\right) > 0. \]

\[ \square \]

**Proposition 7.2**

1. For any \( \Lambda, \Lambda' \in \mathcal{L}(\hat{X}) \) such that \( \Lambda \subset \Lambda' \) we have \( 0 \leq N_{\Lambda}(\hat{\gamma}) \leq N_{\Lambda'}(\hat{\gamma}) \) for any \( \hat{\gamma} \in F \).

2. \((F, \mathcal{B}(\hat{\Gamma}) \cap F)\) is a standard Borel space.

3. For any \( \Lambda \in \mathcal{L}(\hat{X}) \) and all \( n \in \mathbb{N}_0 \) the set \( \Pi^{\text{gc}}_{\Lambda}(N^{-1}_\Lambda(\{n\}), \hat{\gamma}) > 0 \) holds for all \( n \) and all \( \hat{\gamma} \in F \).
4. For each $\Lambda \in \mathcal{L}(\hat{X})$ there exists $\Lambda' \supset \Lambda$ such that $\Pi_{\Lambda'}^{gc}(N_{\Lambda}^{-1}(\{n\}) \cap N_{\Lambda}^{-1}(\{m\}), \hat{\gamma}) > 0$ whenever $\hat{\gamma} \in \{N_{\Lambda'} = m\}$ and $n \geq m \geq 0$.

**Proof.** Since 1 is obvious and 2 is a known result, see e.g., [Shi94], then it remains to prove 3 and 4.

3. Let $\Lambda \in \mathcal{L}(\hat{X})$ and $n \in \mathbb{N}_0$ be given. Then using the same estimate for $W$ as in Corollary 5.4 we have

$$
\Pi_{\Lambda}^{gc}(N_{\Lambda}^{-1}(\{n\}), \hat{\gamma}) = \frac{\Xi_{\Lambda}^{\leq \infty}(\hat{\gamma})}{\Xi_{\Lambda}(\hat{\gamma})} \int_{\Gamma_{\Lambda}} \Pi_{\{N_{\Lambda} = m\}}(\hat{\eta}) e^{-E(\hat{\eta}) - W(\hat{\eta}, \hat{\gamma}_{\Lambda})} \pi_{\Lambda}(d\hat{\eta})
$$

$$
\geq \frac{\Xi_{\Lambda}^{\leq \infty}(\hat{\gamma})}{\Xi_{\Lambda}(\hat{\gamma})} \int_{\Gamma_{\Lambda}} e^{-E(\hat{\eta}) - nC(\hat{\gamma})} \pi_{\Lambda}(d\hat{\eta})
$$

$$
> 0,
$$

where the last inequality is a consequence of the Lemma 7.1.

4. For a given $\Lambda \in \mathcal{L}(\hat{X})$ let $\Lambda' \in \mathcal{L}(\hat{X})$ be such that $\Lambda \subset \Lambda'$. Then if $n \geq m$ we have

$$
\Pi_{\Lambda'}^{gc}(N_{\Lambda'}^{-1}(\{n\}) \cap N_{\Lambda}^{-1}(\{m\}), \hat{\gamma}) = \frac{\Xi_{\Lambda'}^{\leq \infty}(\hat{\gamma})}{\Xi_{\Lambda'}(\hat{\gamma})} \int_{\Gamma_{\Lambda'}} \Pi_{\{N_{\Lambda'} = m\}}(\hat{\eta}) e^{-E(\hat{\eta}) - W(\hat{\eta}, \hat{\gamma}_{\Lambda'})} \pi_{\Lambda'}(d\hat{\eta})
$$

$$
\geq \frac{\Xi_{\Lambda'}^{\leq \infty}(\hat{\gamma})}{\Xi_{\Lambda'}(\hat{\gamma})} \int_{\Gamma_{\Lambda'}} \Pi_{\{N_{\Lambda'} = m\}}(\hat{\eta}) e^{-E(\hat{\eta}) - nC(\hat{\gamma})} \pi_{\Lambda'}(d\hat{\eta})
$$

$$
> 0,
$$

since $0 < \Xi_{\Lambda'}(\hat{\gamma}) < \infty$ by definition of $F$. ■

**Theorem 7.3** For all $\mu \in \mathcal{G}_c(\sigma, V)$ equality (7.75) is equivalent to

$$
\mu(B \cap N_{\Lambda}^{-1}(0)) > 0,
$$

whenever $B \in \mathcal{B}_{\Lambda'}(\hat{\Gamma}) \cap F$ with $\mu(B) > 0$.

I can not prove it even for the case $B = \hat{\Gamma}$ and for the Gibbs measure with an empty boundary condition.

**Proof.** Consider $\Pi_{\Lambda}$ as a specification on $(F, (\mathcal{B}_\Lambda(\hat{\Gamma}) \cap F))_{\Lambda \in \mathcal{L}(\hat{X})}$. Then $\mu$ is a Gibbs measure w.r.t. the new specification iff $\mu \in \mathcal{G}_c(\sigma, V)$ and $\mu(F) = 1$. ■
8 Examples

In this section we give an application of the theory presented before. Namely, we consider as underlying space a principle fiber bundle $M$ with base space $X$, a Riemannian manifold, and a Lie group $G$ as a typical fiber. The canonical projection from $M$ to $X$ is denoted by $p$. On $X$ we consider the Riemannian volume $m$ and the Haar measure $m_G$ on $G$. Since, locally the fiber bundle $M$ is trivial, i.e., for any $x \in X$ there exists a neighborhood $U \subset M$ and a diffeomorphism $\Psi : p^{-1}(U) \to U \times G$, then the intensity measure $\sigma$ on $M$ is given by $m \times m_G$ on each neighborhood of any point of $M$.

We now describe the group of diffeomorphisms on the fiber bundle $M$ to be considered. Before, let us explain the behind. We start from a subgroup of $\text{Diff}_0(M)$ which characterized measures on $M$ via quasi-invariant property, namely if $\tau$ is a measure on $M$ which is quasi-invariant with respect to the considered subgroup, then we show that, locally $\tau$ is up to a constant, equal to $m \times m_G$. Therefore, all the results concerning marked Poisson and canonical marked Gibbs measures are also true in this case.

As before we denote by $\text{Diff}_0(X)$ the group of diffeomorphisms on $X$ with compact support. Let $\phi : M \to M$ be a diffeomorphism such that locally it is given by the left action of $G$ on $M$, more precisely, for each $x \in X$ let $F_x = p^{-1}(x) = \{pg \mid g \in G, p(p) = x\}$ be the fiber over $x$ and $(U, \varphi)$ a local chart in $X$, then ....

**Theorem 8.1** Let $G$ be a locally compact topological group and $m_1, m_2$ left-invariant Radon measures on $G$. Then there exists a constant $C > 0$ such that $m_1 = Cm_2$.

**Proof.** Let $F, H \in C_0(G)$ be given and consider the following integrals

$$\int_G F(g)m_1(dg) \int_G H(h)m_2(dh).$$

Then we can evaluate these integrals, according to Fubini theorem (which is valid since the measures are Radon! ) and the left-invariant property of $m_1$ as

$$\int_G F(g)m_1(dg) \int_G H(h)m_2(dh) = \int_G \int_G F(g)H(h)m_2(dh)m_1(dg) = \int_G \int_G F(hg)H(h)m_1(dg)m_2(dh).$$
On the other hand, consider the integrals

\[ \int_G H(g)m_1(dg) \int_G F(h)m_2(dh) \]

and evaluate them using similar arguments as before but using the left-invariant property of \( m_2 \):

\[ \int_G H(g)m_1(dg) \int_G F(h)m_2(dh) = \int_G \int_G H(g)F(h)m_1(dg)m_2(dh) \]
\[ = \int_G \int_G H(h^{-1}g)F(h)m_1(dg)m_2(dh) \]
\[ = \int_G \int_G H(h^{-1}g)F(h)m_2(dh)m_1(dg), \]

if we assume that \( H(g) = H(g^{-1}) \), then the above equality is gives

\[ \int_G \int_G H(g^{-1}h)F(h)m_2(dh)m_1(dg) = \int_G \int_G H(h)F(gh)m_2(dh)m_1(dg). \]

Therefore we have

\[ \int_G F(g)m_1(dg) \int_G H(h)m_2(dh) - \int_G H(g)m_1(dg) \int_G F(h)m_2(dh) \]
\[ = \int_G \int_G H(h)[F(hg) - F(gh)]m_1(dg)m_2(dh). \quad (8.76) \]

Now let \( F_1, F_2 \in \mathcal{C}_0(G) \) and \( \varepsilon > 0 \) be given. Then according to Lemma 8.3 there exists \( H_{\varepsilon} \geq 0, \rho_{F_i} \geq 0 \) with compact support and

\[ 0 < \int_G H_{\varepsilon}(h)m_2(dh) < \infty \quad (8.77) \]
\[ |F_i(hg) - F_i(gh)| \leq \varepsilon \rho_F(g), \forall h \in \text{supp} H_{\varepsilon}. \quad (8.78) \]

This allow us to obtain the following estimate for (8.76)

\[ \left| \int_G F_i(g)m_1(dg) \int_G H_{\varepsilon}(h)m_2(dh) - \int_G H_{\varepsilon}(g)m_1(dg) \int_G F_i(h)m_2(dh) \right| \]
\[ \leq \int_G H_{\varepsilon}(h)m_2(dh) \int_G \varepsilon \rho_{F_i}(g)m_1(dg). \quad (8.79) \]
We notice that if $\emptyset \neq U \subset G$ is open, then $m_i(U) > 0$. In fact, if $m_i(U) = 0$, then $m_i(gU) = 0$. If we assume that $G$ is $\sigma$-compact, i.e., $G = \bigcup_{i=1}^{\infty} (g_iU)$, then $m_i(G) \leq \sum_{i=1}^{\infty} m_i(g_iU) = 0$ which is a contradiction, since $m_i \neq 0$. This, in particular, implies that $\int_G F(h)m_i(dh) > 0$ iff $F \neq 0$. We proceed by dividing both members of inequality in (8.79) by

$$
\int_G F_i(g)m_1(dg) \int_G H_\varepsilon(h)m_2(dh) > 0.
$$

Then we obtain:

$$
\left| 1 - \frac{\int_G H_\varepsilon(g)m_1(dg) \int_G F_i(h)m_2(dh)}{\int_G F_i(g)m_1(dg) \int_G H_\varepsilon(h)m_2(dh)} \right| \leq \varepsilon \frac{\int_G \rho F_i(g)m_1(dg)}{\int_G F_i(g)m_1(dg)}.
$$

(8.80)

As $\varepsilon$ is arbitrary this implies that

$$
\lim_{\varepsilon \to 0} \frac{\int_G H_\varepsilon(g)m_1(dg)}{\int_G H_\varepsilon(h)m_2(dh)} = \frac{\int_G F_i(g)m_1(dg)}{\int_G F_i(h)m_2(dh)}. \tag{8.81}
$$

and hence

$$
\frac{\int_G F_i(g)m_1(dg)}{\int_G F_i(h)m_2(dh)} = \frac{\int_G F_2(g)m_1(dg)}{\int_G F_2(h)m_2(dh)}.
$$

Lemma 8.2 (Uniform continuity) Let $G$ be a locally compact group and $F \in C_0(G)$. Then, for any $\varepsilon > 0$ there exists an open set $U_\varepsilon \subset G$, such that

$$
|F(hg) - F(g)| < \varepsilon, \quad |F(gh) - F(g)| < \varepsilon, \quad \forall h \in U_\varepsilon, \forall g \in G, \tag{8.82}
$$

or equivalently

$$
|F(h) - F(g)| < \varepsilon, \quad \forall g, h \in G, g^{-1}h \in U_\varepsilon.
$$

Proof. See, e.g., Theorem 3.2.5 in [Lut76].

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Lemma 8.3 Let $G$ be a locally compact group, $m$ a left-invariant measure on $G$ and $F_1, F_2 \in C_0^+(G)$ be given. Then for any $\varepsilon > 0$ there exists positive functions $H_{\varepsilon}, \rho_{F_1}, \rho_{F_2}$ with compact support such that $H_{\varepsilon}(g^{-1}) = H_{\varepsilon}(g)$

$$0 < \int_G H_{\varepsilon}(h)m(dh) < \infty,$$

$$|F(hg) - F(gh)| \leq \varepsilon \rho_{F}(g), \forall h \in \text{supp} H_{\varepsilon}.$$ 

Proof. It is clear that from (8.82) that for $\varepsilon > 0$ exists open set $U_{\varepsilon}$ such that

$$|F_i(hg) - F_i(gh)| < \varepsilon, \forall h \in U.$$ 

Without lost of generality $\overline{U_{\varepsilon}}$ is compact and $U_{\varepsilon}^{-1} = U_{\varepsilon}$. Put $H_{\varepsilon} := \mathbb{1}_{U_{\varepsilon}}$. The set

$$A_i := \overline{U_{\varepsilon}^{-1}} \cup \text{supp}(F_i) \cup \text{supp}(F_i) \overline{U_{\varepsilon}^{-1}}$$

is still compact and put $\rho_{F_i} := \mathbb{1}_{A_i}$.

Acknowledgments

We would like to thank Prof. A.M. Vershik and Dr. V. Liebscher for many useful discussions during the preparation of this work. We would also like to thanks Prof. Ana Bela Cruzeiro, Prof. J.C. Zambrini, Dr. Maria João Oliveira, and Prof. Samuel for the hospitality during a very pleasant stay at GFM of Lisbon University during the conference “Stochastic Analysis and its Applications” in October 2000 where most of this work was realized.
A Quasi-invariant measures on topological groups

Theorem A.1 Let $G$ be a topological group with a left-invariant measure $m$. Let $\mu$ be a non-zero $\sigma$-finite measure quasi-invariant w.r.t. the right action of $G$. Then $\mu$ is equivalent to $m$.

Remark A.2 If $\mu$ is only quasi-invariant w.r.t. to a subgroup $G_0$ of $G$. The following proof is not working.

Proof. Because $\mu$ is $\sigma$-finite we can assume w.l.o.g. that $\mu$ is a probability measure, i.e. let $(G_n)_{n\in\mathbb{N}}$ be a sequence of sets of finite non-zero measure and define

$$\hat{\mu} := \sum_{n=1}^{\infty} 2^{-n} \mathbb{1}_{G_n} \mu(\cdot \cap G_n)^{-1}.$$ 

For every measurable subset $A \subset G$ we have by Fubini

$$\int_G \int_G \mathbb{1}_A(hg) m(dg) \mu(dh) = \int_G \int_G \mathbb{1}_A(hg) \mu(dh) m(dg).$$

According to left-invariance of $m$ we obtain for the left-hand side

$$\int_G \int_G \mathbb{1}_A(g) m(dg) \mu(dh) = m(A) \mu(G)$$

and for the right hand side we notice that $\mathbb{1}_A(hg) = \mathbb{1}_{Ag^{-1}}(h)$ that

$$\int_G \mu(Ag^{-1}) m(dg) = \int_G \int_A \frac{d(\mu g^{-1})}{d\mu}(h) \mu(dh) m(dg).$$

So altogether we have that

$$m(A) \mu(G) = \int_G \mu(Ag^{-1}) m(dg) = \int_G \int_A \frac{d(\mu g^{-1})}{d\mu}(h) \mu(dh) m(dg) \quad (A.83)$$

If $m(A) = 0$ then for $m$-a.a. $g \in G$

$$\mu(Ag^{-1}) = 0.$$

So there exists a $g \in G$ which fulfills the above equation. According to quasi-invariance then the above equation also holds for all $g \in G$ and specially for $g = e$. Thus $\mu(A) = 0$.

Conversely, assume that $\mu(A) = 0$, then by quasi-invariance also $\mu(Ag^{-1}) = 0$ for all $g \in G$. Thus (A.83) implies $m(A) = 0$.  

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Remark A.3 Assume that we have only quasi-invariance for a subgroup \( G_0 \).
Then if \( m(A) = 0 \) we have only that for \( \mu \)-a.a. \( g \in G \)
\[
\mu(gA^{-1}) = 0.
\]
But we do not know if it is true for a single \( g \) from \( G_0 \). So we would need
that \( \mu(gA^{-1}) \) is continuous in \( g \). For the other direction we have an \( A \) with
\( \mu(A) = 0 \). Then
\[
\mu(gA) = \int_A \frac{dg^* \mu}{d\mu}(h) \mu(dh) = 0
\]
for all \( g \in G_0 \). Thus for every \( g \in G_0 \) and for \( \mu \)-a.a. \( h \in G \) we have that
\[
\frac{dg^* \mu}{d\mu}(h) = 0.
\]

Corollary A.4 Let  \( G \) be a topological group with a left-invariant measure \( m \) and a right-invariant measure \( m' \), then \( m \) and \( m' \) are equivalent.

Proof. Define \( I : G \to G, \ g \mapsto g^{-1} \) and denote \( R_g, g \in G \) the right
translation in \( G \). Then we have
\[
\int_G f(h)(R_g \circ I^* \circ m)(dh) = \int_G f(hg)(I^* \circ m)(dh)
\]
\[
= \int_G f(h^{-1}g)m(dh) = \int_G f \circ I(g^{-1}h)m(dh)
\]
\[
= \int_G f \circ I(h)m(dh)
\]
\[
= \int_G f(h)(I^* \circ m)(dh).
\]
So \( I^* \circ m \) is a right-invariant measure.  

Corollary A.5 Let \( \mu \) be a probability measure on a topological group \( G \),
which fulfills the second axiom of countability. Assume that \( \mu \) is quasi-
invariant w.r.t. a dense subset \( G_0 \subset G \) and that the Radon-Nikodym deriva-
tive
\[
g \mapsto \frac{dg \mu}{d\mu}
\]
can be extended to a continuous map from \( G \) to \( L^1(G, \mu) \). Then \( \mu \) is quasi-
invariant w.r.t. \( G \). If \( m \) is a left-invariant measure on \( G \) then \( \mu \) is equivalent
to \( m \).
Proof. Let $f$ be a bounded, continuous function. For any $g \in G$ there exists a sequence $(g_n)_{n \in \mathbb{N}}$ in $G_0$ such that $\lim_{n \to \infty} g_n = g$. Then by continuity

$$L^1(G, \mu) = \lim_{n \to \infty} \frac{d(g_n)_* \mu}{d\mu} = \frac{dg_\cdot \mu}{d\mu}$$

and $\lim_{n \to \infty} f(g_n \cdot) = f(g \cdot)$ pointwisely. Thus together

$$\int_G f(gh) \mu(\text{dh}) = \lim_{n \to \infty} \int_G f(g_n h) \mu(\text{dh})$$

$$= \lim_{n \to \infty} \int_G f(h) \frac{d(g_n)_* \mu}{d\mu}(h) \mu(\text{dh})$$

$$= \int_G f(h) \frac{dg_\cdot \mu}{d\mu}(h) \mu(\text{dh})$$

So for all bounded continuous $f$ we have that

$$\int_G f(gh) \mu(\text{dh}) = \int_G f(h) \frac{d(g)_* \mu}{d\mu}(h) \mu(\text{dh}).$$

The set of all $f$ which fulfill the above equation form a monotone class and thus the equation holds for all bounded measurable $f$. 

Remark A.6 If $\mu$ is not a probability measure we have to go back in some way to the above case. Either we know something more about $G_0$, for example something like that for every $g \in G$ there exists an approximating sequence $(g_n)_{n \in \mathbb{N}}$ and a sequence of set with finite measures, which the $g_n$ uniformly transport not to something of infinite measure. then we can try to reduce to this situation.

Or assume that there exists a continuous, strict positive function $\rho \in L^1(G, \mu)$. Define a probability measure. $\mu' := \rho \cdot \mu$. Then

$$\frac{dg_\cdot \mu'}{d\mu'} = \frac{\rho(g^{-1} \cdot)}{\rho(\cdot)} \frac{dg_\cdot \mu}{d\mu}$$

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References


